JET PROPULSION

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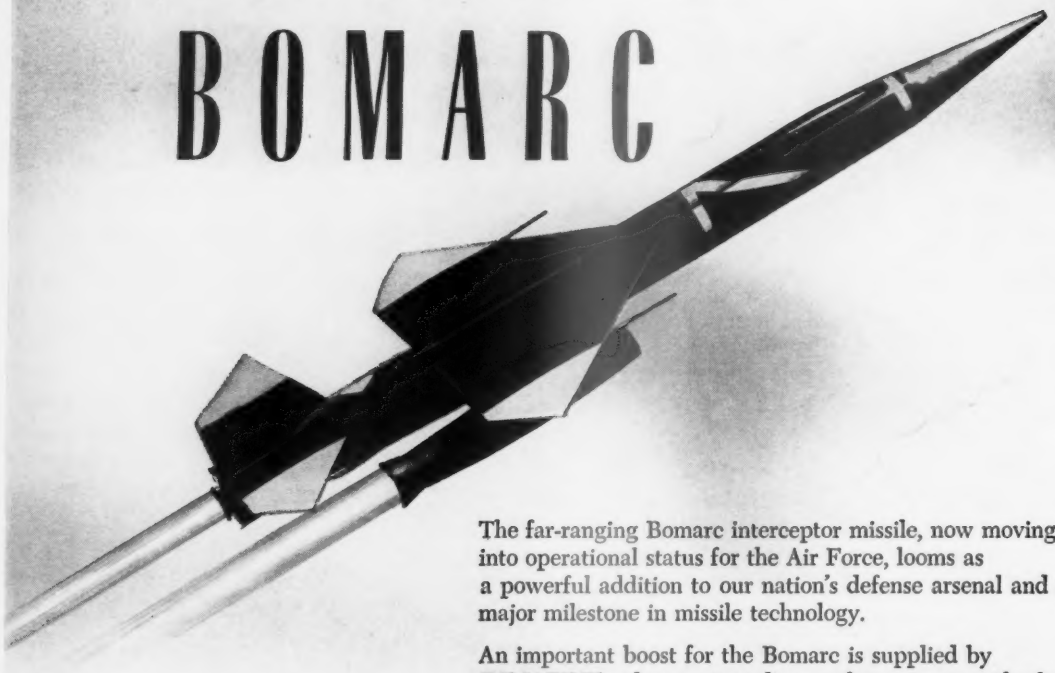

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This Journal is a publication of the American Rocket Society devoted to the advancement of the field of jet propulsion through the dissemination of original papers disclosing new knowledge or new developments. As used herein, the term "jet propulsion" embraces all engines that develop thrust by rearward discharge of a jet through a nozzle or duct; and thus it includes air-consuming engines and underwater systems as well as rockets. JET PROPULSION is open to contributions dealing not only with propulsion but with other aspects of jet-propelled flight, such as flight mechanics, guidance, telemetering, and research instrumentation. Increasing emphasis will be given to the scientific problems of extraterrestrial flight.

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Recent Advances in Astrodynamics

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Robert M. L. Baker Jr., presently a Staff Member, Aeronutronic Systems, Inc. and an instructor of courses in Astrodynamics at the University of California, Los Angeles, obtained a B.A. and M.A. in Physics, and a Ph.D. in Engineering (and Astronomy). His specialties are in the fields of celestial mechanics and rarefield gas-dynamics. Dr. Baker is currently engaged in studies concerning space-vehicle orbits, meteoritics, low density drag and sputtering with particular emphasis on high-speed re-entry from interplanetary orbits.

1 Introduction

A PHENOMENON of the post-Sputnik world has been a spreading of the realization of a few that celestial mechanics has become an engineering subject. But the classical field of celestial mechanics, as delimited by many of its present-day practitioners, is at once too broad and too narrow. It is too broad in that it extends itself into areas such as the structure and motions of the galaxies and even of the stars and the nebulae that are found in them. It is too narrow in that it has not traditionally included forces that are important to the study of trajectories of objects such as those that we are now sending out to space. Accordingly, the new term "astrodynamics" has come to be used at least partly in its stead.

Astrodynamics is defined as including those parts of celestial mechanics, geophysics, aerodynamics, electromagnetic theory, exterior propulsion theory and observation theory that bear on the trajectories of planets, comets, meteorites and artificial vehicles above the denser portions of the Earth's atmosphere. It may be noted that astrodynamics bears the same relationship to astronautics as does aerodynamics to aeronautics.

It is evident that certain of the areas of astrodynamics have been overworked while others have been neglected and await intensive research. In the former category, we list optimization of orbits, satellite stay time, equatorial bulge perturbations, etc. The investigation of these topics has actually been hindered by the variety of notation introduced and the lack of active coordination among the numerous contributors. (It is estimated that there have been published over one hundred unrelated and often redundant intercompany memoranda, technical notes and papers on the subject of equatorial bulge perturbations alone.) On the other hand, articles dealing with precision orbits, with precision study of nongravitational forces, etc., have been conspicuous by their absence from most astronautical literature; and it is in these areas that we will look for the future advances in astrodynamics.

In this connection, one of the most significant of the recent advances in astrodynamics has been the spread of the realization that the subject is a complex one, that not only is there a need for a thoroughgoing knowledge of what has been done in the whole field of celestial mechanics, but also that the advent of artificial vehicles in space is presenting us with new problems that are worthy successors to those solved by such men as Clairaut, Lagrange, Laplace and Gauss. This realization is focused in part on the distinction that we can make between "feasibility orbit work" and "precision orbit work."

Many excellent feasibility studies have been made in recent years. In this country some of the pioneering work may be found in the publications of Rand Corporation, and in this country and abroad in the publications of the various rocket societies, especially the AMERICAN ROCKET SOCIETY and the British Interplanetary Society. A significant amount of work has been done also in proposals for study contracts submitted to military and scientific research organizations and in inter- and intracompany memoranda and technical notes. Feasibility studies are characterized by the search for optimal orbits and minimum requirements upon fuel and guidance. Notable progress has been made in error analyses, but they have tended to lull people who have only a cursory acquaintance with the trajectory field into the comfortable delusion that it is, after all, a simple one. This is because these studies may make use of approximate theories, whether based on the two-body problem or taking into account the perturbations of a third or a fourth body or even of the Earth's bulge. They may be based upon approximate coordinates, such as ones based on the assumption that the moon is moving uniformly in a perfect circle or is fixed in a rotating framework. And finally they may be based upon approximate constants, such as the laboratory value of G the constant of gravitation, which is good to only three significant figures as determined in terrestrial laboratories.

In precision orbits, on the other hand, one must leave no stone unturned in the quest for accuracy in the basic theoretical developments of perturbation theory, in the consideration of nongravitational forces, in the coordinates and reference frameworks, in the constants, in observation and in correction theory. The more recent advances in each of these areas will be explored in what follows.

2 Perturbation Theory

The term "perturbation" is used loosely. Sometimes it is used to designate the sum total of forces acting on an object over and above a principal force that is responsible for some kind of analytic reference orbit or for any one of the components that goes to make up the sum total. It may be used also for the resulting accelerations, or for the departures from a fixed reference orbit or for the variations of parameters that

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define a varying reference orbit. It is evident that these are capable of more specific reference as "perturbative forces," "perturbative accelerations," "perturbative departures" and "perturbative variations." By extension the term is applied also to a process that is not strictly perturbative, but involves numerical integrations of components of the total acceleration, without the use of any reference orbit. The latter process, which has been widely employed in recent years, sometimes without realization of its relationship to the historical development of perturbation theory, is known as Cowell's method. The term "perturbation" also appears in the terms "special perturbations" and "general perturbations," the former referring to the handling of perturbation problems by numerical integration and the latter to the development into series, as of sines and cosines, that can be integrated term by term.

In special perturbations the integration of the departures from a fixed reference orbit is known as Encke's method. The method of variation of parameters, which has been simplified by the work of Herrick (16)[3] and Musen (28), utilizes a reference orbit that varies in such a way as always to agree in position and velocity with the actual path. These two methods and Cowell's method indicate the three basic approaches to the perturbation problem, but it is possible to find many variations or combinations of them, a notable example being Hansen's method. Some criteria for choice between methods of special perturbations are well known; others are emerging; still others are needed. The best established criteria are for choice between Cowell's method and the other two.

Cowell's method is to be preferred when the perturbations are very large. Such, for example, are the perturbations of the sun on a distant satellite, such as the ninth and twelfth satellites of Jupiter; such also might be the case for large thrust perturbations during the modification of an interplanetary orbit. In these circumstances the "perturbative accelerations" would be as large or as unmanageable as the principal acceleration term, on which the reference orbit is based. Cowell's method would accordingly be handled with as short an interval of integration as the more strictly perturbative methods, and its equations, not involving the relatively complex developments of the analytic reference orbit, would be much simpler to compute and program.

When the perturbation terms are very small compared with the principal acceleration, however, Encke's method and the method of variation of parameters have a distinct edge over Cowell's method in that longer intervals may be used in the integration, thus reducing both the amount of calculation and the inevitable accumulation of end-figure errors in the integral. The orbit of the minor planet Icarus, which comes closest of all planets to the sun and so is subject to very large primary accelerations, is a very good example of an integration in which Cowell's method cannot or should not be used (17). Even more questionable is the use of Cowell's method in the integration of the trajectories of lunar vehicles. For these trajectories the perturbations of the sun and moon are almost negligible at the start, and the many steps, at short intervals of time, that are necessary to integrate a Cowell trajectory, because of the rapid variation in the Earth's gravitational attraction, may be avoided by the use of a reference orbit based simply on the two-body problem. A very large number of Cowell integrations of lunar trajectories have nevertheless been ground out of modern computing equipment. Such calculations may be justified in feasibility work when the amount of calculation is of no importance, but when precision requires that the accumulation of error in numerical integration be kept to a minimum, a Cowell integration scheme should be called into question. In one such calculation it is estimated that the accumulation of error amounts to nine significant figures! It constitutes an advance in astrodynamics then, that several organizations are experimenting with Encke integrations in the realm of lunar trajectories.

[3] Numbers in parentheses indicate References at end of paper.

There are several devices by which one may reduce the effect of the perturbations. Doing so may make it possible to ignore the remaining perturbations entirely in feasibility or other approximate studies. One of these is the adjustment of the central mass, or of the constant of gravitation, in such a way as to transfer an average effect of the perturbations into the principal or two-body term. Only the departures from the average effects then remain as perturbations. The average effect of the equatorial bulge, for example, is to augment the central mass by an amount depending upon the inclination of the orbit. In the n-body problem, the average effect of objects inside the orbit under study is to increase the central mass by an amount somewhat in excess of their own masses. Exterior objects decrease the average pull of the central mass by amounts considerably less than their own masses.

Another possible way to reduce the perturbations is to adjust or alter the analytical reference orbit. Acting on this suggestion Baker (2) has shown that a rectilinear gravity-free drag orbit is preferable as a reference orbit in certain re-entry problems. Sterne (34) and Garfinkel (12) have studied interesting solutions to the satellite problem, when there are bulge perturbations present, by employing an exactly soluble Hamiltonian. Unpublished work indicates that there are special or approximate solutions of the three-body problem of the same character. For feasibility work, as indicated above, these solutions may provide us with useful information that is relatively complete. For precision work, however, it must be remembered that at best they are approximations to the real circumstances. They may supply a useful reference orbit for Encke-type perturbations or for the variation of parameters, but additional perturbation developments, as carried out in (2), must be undertaken for the full solution of the problem. Accordingly the reduction in the complexity of the perturbation work must be balanced against the complexity of the reference orbit. If the reference orbit is exceedingly complex, involving functions that are difficult to compute or tabulate, the new reference orbit may involve more complications than it eliminates in the perturbations. The three-body problem has been disappointing in this respect; see (23 and 26). Analytical solutions are very lengthy and would probably involve impractical calculation and an inordinate number of terms. The simple integral named for Jacobi has been used effectively in checking perturbation integrations and may in the future serve a useful purpose in the variation of parameters.

The satellite problem has proved to be immeasurably more complex than it was generally conceded to be before Sputnik. Numerical integration is unsatisfactory because of the accumulation of error. Series integration is unsatisfactory because of the excessive number of terms that must be carried. Further experimentation is to be highly desired. It should include such matters as the investigation of Tchebycheff and other polynomials in series integration, modification of the equations of motion, modifications of the reference orbit, and combinations of series integration with numerical integration.

Ion or other low-thrust propulsion offers an interesting challenge to perturbation theory. The continuous thrust is easy enough to handle theoretically, see for example (35), but it may be considerably smaller than other perturbative forces, and so may become lost in the uncertainties of the latter. Thus for a take-off from a circular satellite orbit the design trajectory may be a uniform spiral, whereas the actual path will probably develop a sort of "elliptic resonance," developing into a series of intertwined loops from which the "break-away" direction will be highly uncertain. Paradoxically, this circumstance will be less troublesome for interplanetary voyaging than for trips to the moon. For the latter it is desirable, to put it mildly, to go somewhat in the direction of the moon rather than a contrary direction. For the former, when the vehicle starts to recede from the Earth it will begin to slow down so markedly that, whatever the direction, it will be traveling on a heliocentric orbit with approximately the same velocity and very nearly the same shape as the orbit of the Earth. Ac-

cordingly, subsequent low thrust will follow a pattern very much the same, whatever the direction of the vehicle from the Earth. Once free from the Earth, however, a heliocentric elliptical resonance may prove troublesome.

3 Nongravitational Forces

Only recently have investigators been stimulated to look closely at nongravitational forces in space-vehicle motion. Unfortunately the investigations stem more from attempts to explain away erroneous results of purely gravitational orbit determination than from basic interest in the subject for its own sake. The principal nongravitational forces that appear significantly to affect an object in space are aerodynamic drag, electrical and electromagnetic forces, meteoritic drag, radiation pressure and the relativistic effects. (The last effects, although strictly speaking not nongravitational, are placed in the nongravitational category for convenience.)

Although aerodynamic drag is small for an orbit above the denser atmospheric levels of the Earth and planets, it must be treated with considerable care as a perturbation. Recent investigations of the drag coefficient at high altitudes (4, 27 and 15) have all shown that the drag coefficient is significantly less than the so often employed value of 2. In fact, the transitional parameter (the reciprocal of a modified Knudsen number, α or B) has the value of approximately 0.4 for a satellite at a perigee altitude of 200 km (and with a 1 m radius, surface temperature of 300 K and an orbital semimajor axis of 2 Earth radii) and, consequently, reduces the drag coefficient by approximately 5 to 10 per cent. This recent advance in the aerodynamic side of astrodynamics should soon find employment in satellite orbit determination.

Once in orbit, the space vehicle will acquire an electrical charge that may interact with the Earth's or sun's magnetic field and perhaps with a highly ionized atmosphere, to produce both dissipative and nondissipative forces. The factors which influence the satellite's charge are the photoelectric effect, ion pickup and ion conduction. The resulting charge appears to be quite small, but the forces (e.g., Lorentz, induced drag, etc.) resulting from this charge could become important as perturbations. Pioneering investigation of these electrical forces have been carried out by Howard Chang and Myron Smith of the Rand Corporation and by the authors listed in (14 and 21). In addition to such forces, one might conceive of a magnetohydrodynamic "shock wave" that would affect the motion of a space vehicle. More investigation of these areas is certainly indicated.

Meteoritic drag, occasioned by the momentum transfer of micro-meteorites impacting on the surface of a space vehicle, would be quite similar in action to aerodynamic drag. The question of the density of meteoritic material in space must be resolved, therefore, before adequate conclusions can be drawn from satellite observations. A large amount of theoretical work in this regard can be found in the numerous papers of Whipple and of Kallmann. Typical of the analytical work involved in the interpretation of micro-meteorite detectors on board a space vehicle is the less well known work of Bauer (5). In this connection the problem of surface pitting, not only occasioned by micro-meteorites but also by impinging molecules (sputtering) needs careful attention. The phenomenon of molecular sputtering as studied by Wehner, Henschke, Honig and others has interesting effects, not only on surface deterioration, but also on drag and heat transfer, as noted in (2).

Radiation pressure will be particularly influential in the analysis of the orbital motion of low mass and large area bodies, such as dust particles, balloons, etc. The analysis of such bodies ranges from the classical work of Poynting and of Robertson (32) (done in connection with the study of interplanetary dust) to the more recent investigations of Garwin (13).

Relativistic effects are not only of importance in the motion of the perihelion of Mercury (11, 37), but also in orbital studies of artificial asteroids (38) and satellites of Earth (25). The problem of the efficient incorporation of Einstein terms into perturbation theory is of great importance and deserves careful future investigation.

4 Differential Relationships

Differential formulae are useful in the improvement of an approximation to an orbit into a better approximation, in the correction by thrust of a physical orbit that does not intercept an objective into one that does, and in error analyses or studies of the propagation of uncertainties in initial conditions to uncertainties in terminal conditions. In astronomical work, heretofore, it has been customary to obtain the necessary formulae by differentiation of the basic two-body formulae, usually without any attempt to determine the effects of these changes on the perturbations. Modern electronic calculating equipment, however, has given us a powerful new tool for such analyses. Instead of differentiating the basic formulae, we are now inclined to repeat the basic calculations once for each of the constants that is to be altered in the correction or whose uncertainties are to be studied. From the differences in the end results that correspond to arbitrary alterations in the basic constants, made one at a time, the differential coefficients are determined. These can as well as not include the effect on the perturbations.

The differential correction of orbits is not, as some have supposed, a matter of determining from observation at an instant all of the components of position and velocity. It involves instead the calculation of what an observation would be if the adopted elements were correct. These elements may be the six customary elements of a two-body orbit, or the six components of an initial position and velocity, or some other appropriate set of constants. The discrepancies between calculation and observation are called residuals; it is these that are related differentially to the desired corrections of the starting elements. It follows, then, that if there are six constants to be corrected, a minimum of six residuals is needed. These residuals may be of any accurately observed quantities or of any combination of observed quantities, subject of course to special circumstances that may lead to indeterminateness in the solution.

Rapidly moving Sputniks offer a special challenge to differential correction theory, especially in view of the difficulty of getting an adequate perturbation theory. For finding ephemerides, at least, it may prove to be preferable to adopt a program of continuous correction, based on observation (1), instead of developing a thoroughgoing perturbation theory. Without perturbations or with an insufficient perturbation theory, however, there is a limit to the usefulness of the earlier observations as compared with the later observations. Since the early observations have an important bearing on the accurate determination of the period and semimajor axes, however, they cannot be completely discarded. It is necessary to develop a system in which their effect is properly weighted. An interesting essay in this direction, of a preliminary and theoretical rather than practical character, was presented by Swerling of the Rand Corporation to the Los Angeles "Astrodynamics Colloquium" in May 1958.

5 The Determination of Orbits

In classical astronomical theory it has been customary to determine circular orbits from two observations of direction only or from one observation of the two direction angles and their velocities, and to determine elliptical orbits from three observations of the two direction angles. The circular orbit procedures appear to be directly applicable to the determination of Sputnik orbits, but the elliptical orbit determination procedures are unlikely to offer much to the problem, because the slight departure from great-circle motion usually encountered requires accuracy of observation beyond that which seems likely in satellite observations.

The loss of the classical methods for the determination of elliptical orbits, however, seems to be made up for by many new techniques such as those employed in (20 and 36) as well as those discussed in the references compiled by Benton (6). Most of these are approximate in character, ranging from rough estimates as to when the satellite first passes over an expected point to determinations from radio Doppler observations. The state of orbit determination, however, appears to be somewhat chaotic, and it is questionable whether it will function well for an unexpected and unannounced satellite.

6 Requirements on Observation

The comparative precision of the various types of observation used with satellite and space-vehicle orbit determination is understood somewhat better now than it was a few months ago. The accuracy to be sought in visual observations should preferably be not less than 2 sec (one part in 10^5) for close, fast moving satellites, and should range down to one part in 10^6 or even better for an object at 5 Earth radii. The 2 sec accuracy has been claimed for the ballistic cameras of the Cape Canaveral chain, see (8), and has probably been achieved also with wide angle photographic telescopes at astronomical observatories. The stituation with the yet untested and long delayed Baker-Nunn cameras is not clear; it may be that 1 min is a more realistic estimate for these and perhaps even for the other cameras mentioned.

Two seconds of arc (10^{-5} radians) nevertheless stands as a challenge to the electronic methods of observation. Observations of lesser accuracy will be unlikely to have much value in the improvement of our knowledge of the gravitational environment. For the present, accordingly, we shall tend to discount radio Doppler observations (22) and to hope that transponder-type observations of high accuracy will be planned increasingly.

The question of observation is often associated with the availability of a stable satellite platform in space. In this area of astrodynamics the work of Roberson, Klemperer and others (30, 31, 24, and 3) has been particularly interesting and useful, not only with application to satellite stability, but also with respect to lunar librations.

It is important to distinguish between the processes of observation reduction and orbit determination. Specifically, an observer or observation group should not consider it necessary or even desirable to combine incomplete observations to determine a position. By incomplete observations we mean measures of one or two components of direction without distance, or of distance without direction. With such observations, observers unacquainted with astronomical techniques tend to believe that it is necessary to triangulate in order to obtain a position. Such a position, however, may multiply the uncertainties of the individual observations to the point of vitiating their value. It is far better to supply the orbit analyst with the individual observations, not necessarily in groups capable of triangulation. He will then be able to "represent" the observations, i.e., calculate what they would be if this assumed orbit were correct. The residuals or discrepancies between the observed and the computed values then supply him with a basis for the correction of his assumed orbit (compare section 4).

Accordingly, we may see that any observed datum, a distance (or "slant range"), a direction angle, a radial velocity (or "slant range rate"), etc., can be used by the orbit analyst.

A corollary to the foregoing discussion is that the observer need not attempt to find a zero point, as on a Doppler curve, from the gross values of points on the curve. The orbit analyst instead can compute residuals, and either determine the zero point to their lesser range, or leave it as an unknown for his final solution.

The proper province for observation reduction then is the correction of raw data for known systematic errors, instrumental ones, such as level and azimuth and collimation, or environmental ones, such as refraction and aberration. Even geocentric parallax, especially since it is so large, should be left to the orbit analyst.

7 Coordinate Systems

The observations themselves do not have to be reduced to a common reference system, but the systems in which they are made must be closely tied to one. Observed coordinates are necessarily "topocentric," i.e., referred to the place ("topos") of observation; their frame of reference may be tied to the local horizon or to the equator. In either circumstance the coordinates of the observer (latitude, longitude, height above sea level) and the time must be accurately known and reduced to a specified reference spheroid. The connection between the observation frame of reference and the geocentric dynamical frame of reference is then handled partly in the reduction of observational data (as in topocentric coordinates or direction cosines) and partly in the orbit calculation.

The dynamical frame of reference is usually not an inertial one, especially for geocentric orbits, since relative motion can be handled as easily as inertial motion. But the reference axes should be unaccelerated; the logical choice for such axes is equatorial and equinoctial, i.e., with the plane of the equator adopted for the xy-plane, and the x-axis directed to the vernal equinox. Even in these circumstances, because of precession and nutation, we must choose between several equators and equinoxes:

1 Fixed to such a timeless frame as the mean equator and equinox of 1950.0.

2 Fixed to the mean equator and equinox of the beginning of the year.

3 Fixed to the mean equator and equinox of some arbitrary "epoch," perhaps a date during or shortly before flight.

4 Moving with the mean equator and equinox of date.

5 Moving with the true equator and equinox of date.

6 Fixed to the true equator and equinox of some arbitrary epoch.

The first three, all inertial, differ by a set amount of the "precession of the equinoxes"; the fourth and fifth are noninertial, being affected by progressive precession 3 to 4, and nutation, 4 to 5. System 6 is also inertial, differing from system 3 by a fixed amount of nutation. The orbital work, then, would have to be done in such a frame as 1, 2, 3 or 6; but unfortunately the observer and possibly his reference axes are necessarily in system 5. For a lunar vehicle restricted to a few days in orbit, it may be preferable to work in system 6, neglecting its departures from system 5, but for a long term satellite it will be necessary to consider the problems of precession and nutation with great care.

When observations are made with altazimuth instruments, such as ballistic cameras, and the angular coordinates are read from graduated circles, the full effects of refraction, precession and nutation must be taken into account. When they are made against a stellar background, by measurement of a photographic plate, some of these effects are absorbed by the star-place reductions, but at least the geocentric coordinates of the observer must be corrected for precession and nutation.

8 Astronomical and Gravitational Constants

The problems associated with accurate constants may be illustrated in lunar-flight trajectories by the fact that the best values of the geocentric gravitational constant and the lunar parallax as found by Herrick, Baker and Hilton (18) lead to uncertainties of the order of 100 miles at lunar impact. It is clear that the constants will have to be improved for future spot landings designed to supply depots or expeditions. It is also clear that precise observations of early flights should contribute to the improvement of the basic constants.

For flights to Mars and Venus even greater uncertainties

arise because of the inexactness of the solar parallax, which is essentially the ratio of the Earth's equatorial radius to the distance of the sun (the astronomical unit). It is best to localize this effect in the conversion of the burnout velocity from ft per sec to astronomical units per day; it is easy then to show that the uncertainty in a purely ballistic trajectory at Mars or Venus would be of the order of several hundred thousand miles. Having made the shift to the heliocentric system of units and constants, however, it is possible to adjust the ballistic trajectory and cancel out much of the error in the constants by a program of observation and corrective thrust. This correction can be carried out, because the observations are already, in effect, in the heliocentric system.

The basic data on which the gravitational constant is based were added to critically and effectively by the recent achievement of the Army Map Service enclosing two latitude arcs in this hemisphere and in the other (9). From these measures is determined an excellent value of the Earth's equatorial radius. Joined with the equatorial radius in the determination of the constant of gravity is the acceleration of gravity at the equator. The accuracy of the latter requires both a careful study of the statistical determinations of the acceleration of gravity at the equator from relative observations made around the Earth, and of the basic determination of the acceleration of gravity in Potsdam in 1906. The latter determination has been subject to correction by later determinations at Washington (19) and at Teddington (10). Other accurate determinations are known to be in process.

The distance of the moon has not been determined by parallax measures since the series of observations made in the early part of this century at Greenwich and the Cape of Good Hope; a more recent determination is reported by O'Keefe and Anderson. These observational determinations are somewhat inferior to the determination dynamically, that is, from the constant of gravitation and the period of the moon.

The critical determination of the solar parallax has been subject to a number of painstaking investigations in recent decades. The best value to date seems to be that of Rabe (29).

Recent years have seen a revolution in the accurate determination of the fundamental time variable. The irregularities in the rotation of the Earth have been tied down with increasing accuracy (7), and the result has been the establishment of a new kind of time, more nearly Newtonian, which is to be called Ephemeris Time. It will be the basis of the tables of the sun, moon and planets to be published in the National Ephemerides beginning with 1960. The correction of Greenwich time to Ephemeris time is well presented in Sadler (33).

9 The Future

The current great upsurge of interest in astrodynamics is attested by the many groups being founded in the missile industry to study trajectories. The number of rockets sent theoretically to the moon is beginning to rival the number of persons killed in mystery literature. In some instances the calculation of a single trajectory to the moon fills the heart of its calculator with perhaps too much of a sense of achievement and knowledge. But in the great majority of instances the young men who are assigned to this kind of work soon discover the vastness of the field and tasks before them. They are the men who will make great contributions in the field of celestial mechanics in the future. Some of them have already begun to do so.

The universities are beginning to rise to the challenge presented to them by the increase of interest in this field. We venture to describe the U.C.L.A. program, in part because of our familiarity with it, and in greater part because it can claim priority in the field. Formally, the program may be said to have started with the introduction of the course "Astrodynamics and Rocket Navigation" in 1946, although older courses in the determination of orbits and the reduction of observations are now an integral part of the program. To these have been added four graduate courses. The first master's degrees have been granted under the program this year and, in conjunction with the Department of Engineering, the doctorate. Its program of research dates back to 1937. Through the Institute of Navigation, it sponsors the Astrodynamics Colloquium, which every three weeks brings together about fifty persons from local research institutions and the missile industry for discussions on matters concerned with precision trajectory work. Cooperation with industry is regarded as an essential part of the program, and it is recognized that the future of celestial mechanics is in the hands of a much broader group than was ever the case previously. In return, industry has come to recognize astrodynamics as an essential specialization, no longer to be touched upon lightly without appreciation of its basic complexity.

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An Experimental Study of High-Frequency Combustion Pressure Oscillations[1]

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The paper reports the results of some experiments conducted for investigating the influence of the length of the combustion chamber, the steady-state chamber pressure, the shape of the nozzle, and the equivalence ratio on high-frequency combustion pressure oscillations. The experiments were conducted with a rocket motor burning premixed gases, thereby simplifying the experimental work and eliminating the effects of such variables as atomization, vaporization, mixing, etc. The premixed gaseous propellants used were superheated propane and air, ethane and air, methane and air, hydrogen and air, and ethylene and air. For the longitudinal mode of combustion pressure oscillation the frequencies ranged from 570 to 1750 cycles per sec. The results indicate that a relationship exists between the amplitude and frequency of the combustion pressure oscillations, the combustion chamber geometry, and the burning rate of the propellants.

Introduction

AN EXPERIMENTAL study of some of the factors affecting combustion pressure oscillations has been initiated at the Jet Propulsion Center, Purdue University, with the object of determining the effects of certain pertinent variables upon those oscillations. The research is in its initial phases and is being conducted with a rocket motor burning premixed gases as the propellants. In this manner several of the variables which ordinarily enter into the combustion process for a liquid propellant rocket motor, such as atomization, phase change, mixing, etc., are eliminated.

The initial phase of the experimental program is concerned with the studies of the influence of the following variables when burning a given set of premixed gaseous propellants:

1 Length of the combustion chamber for a fixed diameter.
2 Shape of the converging portion of the exhaust nozzle.
3 Steady-state combustion pressure.
4 Equivalence ratio for the propellant combination.
5 Diameter of the combustion chamber for a fixed length.
6 Location and method for injecting the propellants.

The discussions in the present paper are limited to the experiments concerned with studies of the influence of the following upon the frequency and amplitude of high-frequency combustion pressure oscillations:

1 The length of the combustion chamber for a fixed diameter.
2 The shape of the subsonic part of the nozzle.
3 Variation in the steady-state combustion pressure.
4 Variation in the equivalence ratio.

To date, investigations have been conducted with the following propellant combinations: Premixed superheated propane and air, premixed ethane and air, premixed methane and air, premixed hydrogen and air, and premixed ethylene and air.

Experimental Apparatus

Fig. 1 presents a cross-sectional drawing of the research rocket motor. The internal diameter of the cylindrical combustion chamber is $3\frac{3}{8}$ in. Its length was varied by changing the number of uncooled cylindrical sections. The gaseous fuel and air were introduced into the mixing chamber through concentric holes; the fuel entered through the central tube and the air through the annulus surrounding the latter. The fuel and air were thoroughly mixed by causing the combustible mixture to flow through a screen and around several baffles located in the mixing chamber. The injection system was designed so that it operated with a pressure drop of approximately 150 psi thereby suppressing any tendency for low-frequency combustion pressure oscillations to occur.

The pressures in the combustion chamber were measured with water-cooled capacitance type pressure transducers

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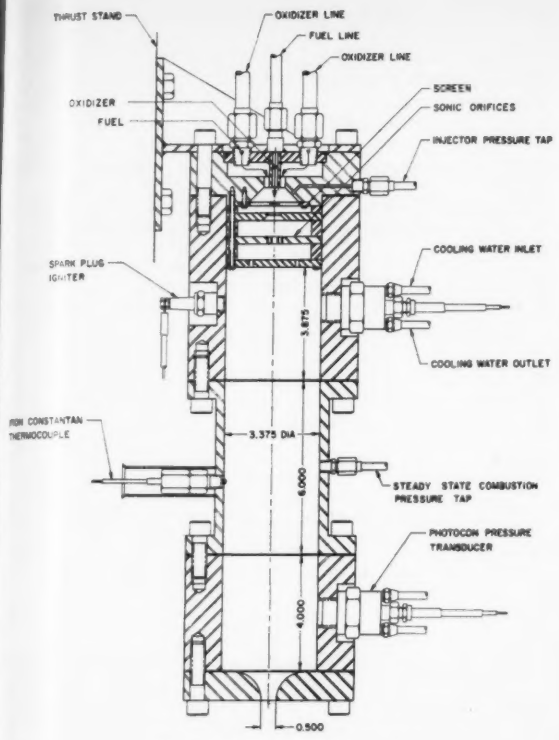

Fig. 1 Research rocket motor

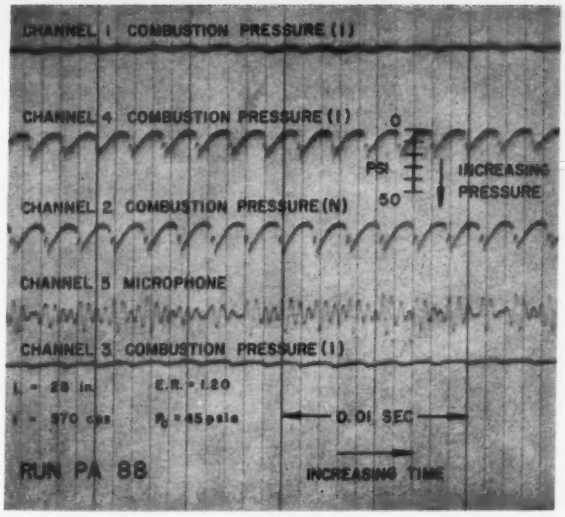

Fig. 2 Oscillograph record illustrating oscillations of a shock type

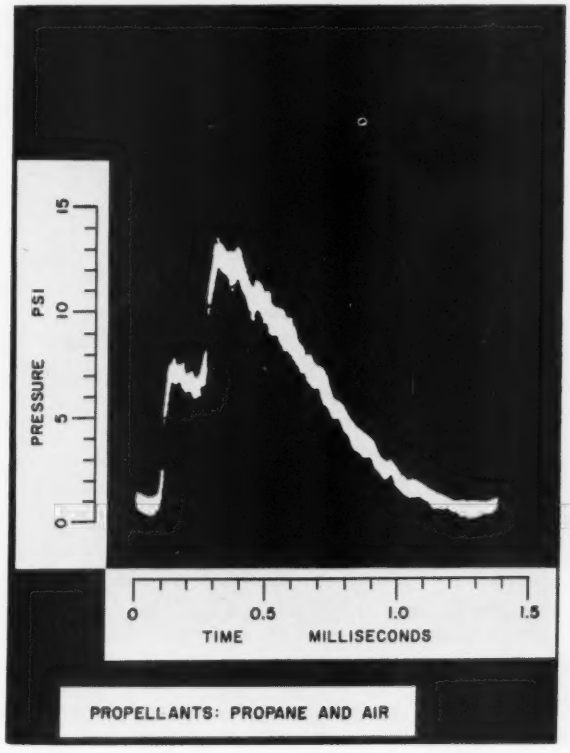

Fig. 3 One cycle of the shock type oscillation

(Photocon Model 342) and their associated electrical equipment (Dynagage Photocon Model DG-101 and Power Supply PS102). The electrical outputs of the transducers were recorded on a six-channel, cathode-ray oscillograph (Hathaway Model SC-16B).

The Photocon transducers were calibrated statically before each experiment. Photocon transducers which were to be operated with water-cooling during an experiment were also water-cooled during their calibrations. The calibration pressures and the combustion pressures measured during an experiment were recorded on the same oscillograph record. The dynamic characteristics of each of the Photocon transducers employed in the experiments and also their associated electrical equipment were determined by the shock tube technique described in (1 and 2).[4]

Experimental Results

The results are presented under the subheadings: Classification of the Oscillations, Effect of the Chamber Length, Effect of the Nozzle Shape, and Effect of the Steady-State Combustion Pressure and Equivalence Ratio.

Classification of the Oscillations

The combustion pressure oscillations of a longitudinal mode have been classified into two principal categories called for convenience the "shock" type and the "sinusoidal" type. Other intermediate categories exist. The shock and sinusoidal types correspond to the oscillations having large and small amplitudes respectively.

Fig. 2 is an oscillograph record illustrating the shock type of oscillation, so termed because the extreme rapid rise in pressure at the beginning of each cycle is quite similar to that for a normal shock wave (1).

Fig. 3 illustrates one cycle of the shock type for a gaseous bipropellant rocket motor burning superheated propane and air. It can be seen that the overall pressure rise comprised two rapid increases in pressure. The first occurred when a longitudinal pressure wave moving toward the injector passed the pressure sensing diaphragm of the Photocon transducer. The second pressure rise was due to the same pressure wave passing the pressure transducer after being reflected from the face of the injector. The extremely short time in which the pressure increases occurred indicates that the pressure wave was substantially a shock wave.

[4] Numbers in parentheses indicate References at end of paper.

Fig. 4 is an oscillograph record illustrating the sinusoidal type of high-frequency combustion pressure oscillations obtained with the gaseous bipropellant rocket motor.

Fig. 5 illustrates a transformation from steady burning to the shock type of oscillation when burning premixed superheated propane and air. Similar transformations were observed for the other propellant combinations investigated.

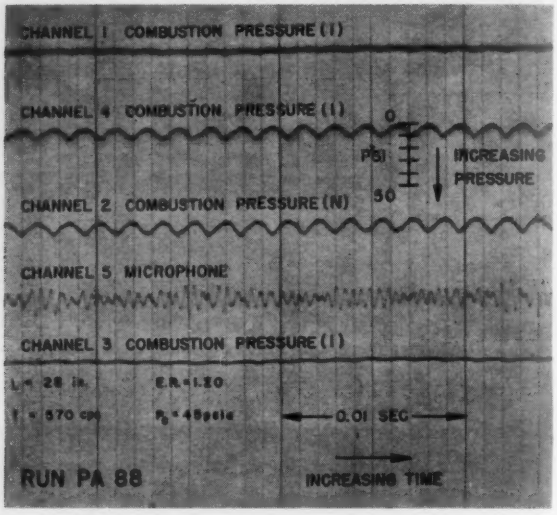

Fig. 4 Oscillograph record illustrating oscillations of a sinusoidal type

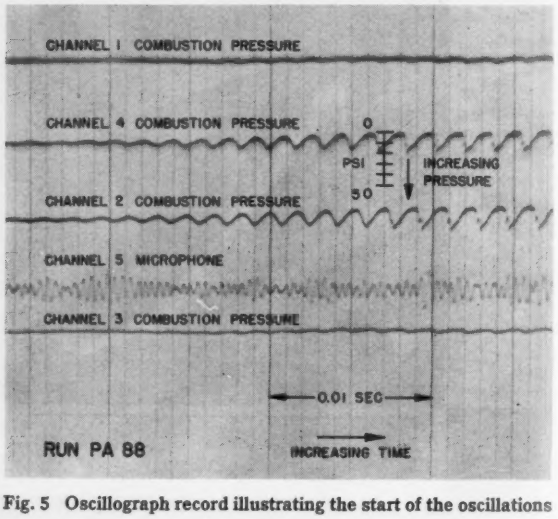

Fig. 5 Oscillograph record illustrating the start of the oscillations

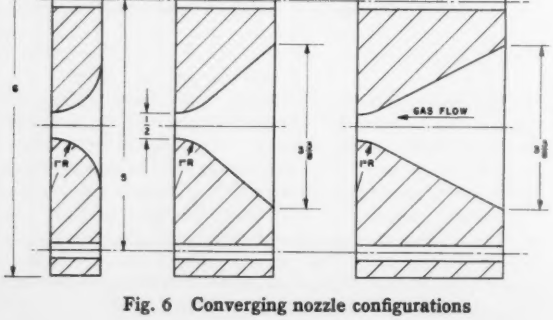

Fig. 6 Converging nozzle configurations

Effect of the Chamber Length

Table 1 presents typical data obtained from experiments with superheated propane and air as the propellants. The data illustrate the effect of changing the combustion chamber length, the only variable in the series of experiments.

Table 1 Typical reduced data from the study on the effect of chamber length for superheated propane and air

Chamber length in.	Type of oscillation	Frequency of the oscillations cps	Amplitude of the oscillations psi
28	shock (L)*	570	22
22.3	shock (L)	685	22
16.3	shock (L)	948	17
12.6	sinusoidal (L)	1219	4

Values for the variables:
Equivalence ratio, 1.2 ± 0.08
Propane line pressure drop, 158 ± 8 psi
Air line pressure drop, 150 ± 10 psi
Mean combustion pressure, 45 psia ± 2 psi

* L indicates a longitudinal mode of oscillation.

The data obtained from experiments with other propellant combinations indicated that varying the length of the combusion chamber influences the amplitude and frequency of the combustion pressure oscillations in a similar manner.

Gaseous bipropellant rocket motors burning premixed superheated propane and air and having combustion chamber lengths shorter than 7 in. did not exhibit high-frequency combustion pressure oscillations of a longitudinal mode. For those premixed gaseous propellants, a length of 7 in. may be considered to be the critical length below which no high-frequency combustion pressure oscillations of a longitudinal mode occur. Table 2 presents the measured critical lengths for four of the propellant combinations that were investigated.

Table 2 Tabulated data of critical lengths*

Oxidizer	Fuel	Critical length in.
air	superheated propane C_3H_8	7
air	ethane C_2H_6	8
air	methane CH_4	11
air	ethylene C_2H_4	6

* The parameters held constant were: (1) steady-state combustion pressure (45 psia), (2) nozzle shape, (3) equivalence ratio.

Effect of the Nozzle Shape

Fig. 6 illustrates the shapes of the three nozzles employed for determining the influence of the shape of the entrance section (subsonic portion) of the nozzle upon the amplitude and frequency of the combustion pressure oscillations. Based upon their axial length the three nozzles will be designated as the 1 in., 2 in., and 4 in. nozzle, respectively.

Figs. 7 and 8 present typical portions of the oscillograph records of the combustion pressure oscillations obtained in experiments in which the only variable was the length of the subsonic portion of the nozzle. The propellant combination was premixed methane and air.

Fig. 7 is an oscillograph record obtained with the 1 in. nozzle. Channel 2 presents the static pressure as measured by a Photocon transducer located 2 in. upstream from the entrance

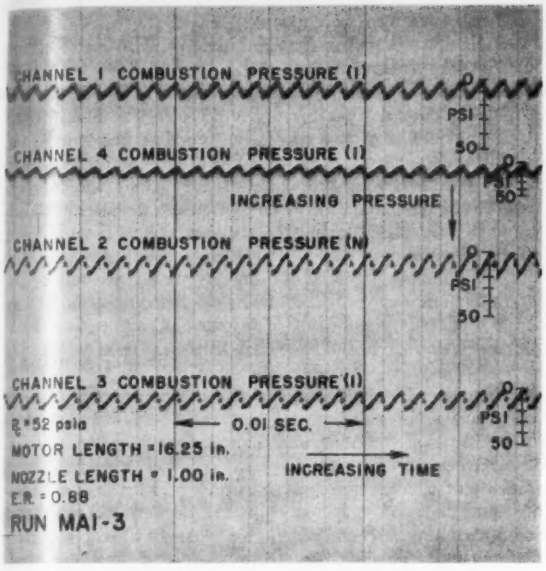

Fig. 7 Oscillograph record for methane and air

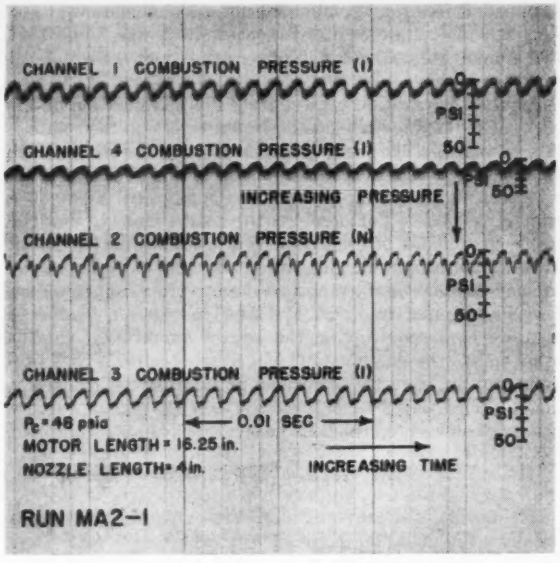

Fig. 8 Oscillograph record for methane and air

to the nozzle. Channels 1, 3 and 4 present the static pressures in the plane 2 in. downstream from the injector face. All of the static pressure records (channels 1, 2, 3 and 4) indicate that the combustion pressure oscillations were of the shock type.

Fig. 8 is an oscillograph record obtained from an experiment with the 4 in. nozzle. The wave shape of the static pressure record obtained from channel 2 (the Photocon transducer located near the nozzle entrance) differs markedly from that of the shock type. In addition, the corresponding pressure records obtained from channels 1, 3 and 4 (the Photocon transducers located near the injector face) also have wave shapes that are different from the shock type. The difference may be attributed to the fact that the entrance section of the nozzle no longer affords a surface from which pressure waves can be reflected in the longitudinal direction with full intensity but causes the reflected pressure waves to interfere with each other.

The Effect of the Steady-State Combustion Pressure and Equivalence Ratio

In the investigation concerned with the effect of the steady-state combustion pressure the latter was varied from 25 to 130 psia, approximately. Fig. 7 is a portion of the oscillograph record obtained with a steady-state combustion pressure of 52 psia when burning premixed methane and air at an equivalence ratio of 0.88. The record indicates that the combustion pressure oscillations were of the shock type; their frequency is 920 cps, and the peak to peak amplitude is 20 psi.

For the same premixed propellants burned at the same equivalence ratio but with a steady-state combustion pressure of approximately 130 psia the combustion pressure oscillations were of the shock type; their frequency was 1000 cps, and the peak to peak amplitude increased to 35 psi.

From additional data similar to those presented in Fig. 7 it is concluded that increasing the steady-state combustion pressure increases the frequency of the combustion pressure oscillations and also the peak to peak amplitude of the oscillations. The aforementioned effects were noted for all of the gaseous bipropellant combinations investigated.

During the course of the investigation of the effect of the steady-state combustion pressure on the combustion pressure

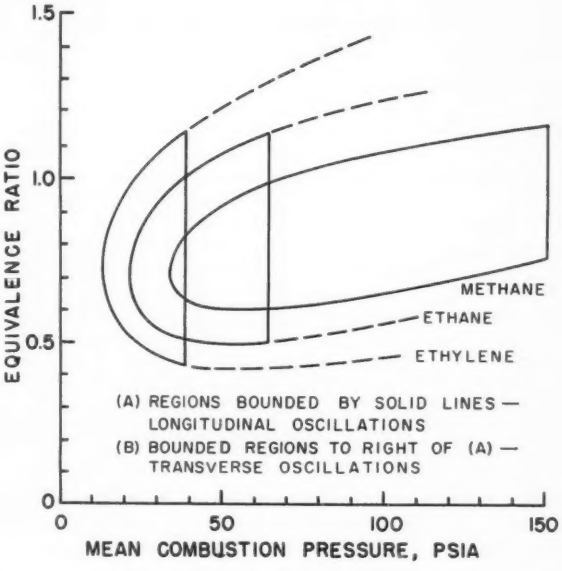

Fig. 9 Stability regions

oscillations, no combustion pressure oscillations were observed when the premixed methane-air propellant combination was burned at equivalence ratios above 1.3, regardless of the magnitude of the combustion pressure, over the range 25 to 130 psia. It was decided, therefore, to investigate the behavior of the other premixed gaseous propellant combinations at different equivalence ratios. Although that investigation has not been completed, the results obtained to date are of sufficient interest to be presented at this time.

Fig. 9 presents regions in which high-frequency combustion pressure oscillations occurred for the premixed propellants methane and air, ethane and air, and ethylene and air. In the latter figure the curves do not represent sharp lines of demarcation between stable and unstable burning, but rather the loci of midpoints of zones wherein the combustion pressure oscillations changed from a high amplitude shock type to a low amplitude sinusoidal type, and eventually into smooth

combustion free from combustion pressure oscillations. For a given steady-state combustion pressure it was found that the aforementioned zones could spread over 0.10 units of equivalence ratio.

Another observation made during the investigation of the effect of the steady-state combustion pressure on the combustion pressure oscillations was that increasing the steady-state combustion pressure caused the mode of the pressure oscillation to shift from a predominately longitudinal mode to a transverse mode. Furthermore, the results indicate that the transverse modes of oscillation close to the injector are of much larger amplitude than those near the entrance to the exhaust nozzle. Of the premixed gaseous propellant combinations investigated, hydrogen and air gave transverse modes of pressure oscillation having the largest amplitude. On the other hand, the combination of premixed superheated propane and air gave the lowest amplitude for the transverse mode. Since this phase of the investigation is still in progress, the results presented should be regarded as being incomplete.

Discussion of Results

The results of the investigations presented in the foregoing are discussed under the subheadings: The Effect of the Combustion Chamber Length, The Effect of the Nozzle Shape, and Effect of the Steady-State Combustion Pressure and Equivalence Ratio.

The Effect of the Combustion Chamber Length

From the investigation of the effect of combustion chamber length it is apparent that the amplitude of high-frequency combustion pressure oscillations ranges from a low amplitude sinusoidal type illustrated in Fig. 4 to the high amplitude shock type illustrated in Fig. 3. For intermediate amplitudes the wave shape is substantially triangular with the changes in pressure, increasing and decreasing, occurring in approximately the same time interval.

Examination of the combustion pressure records for the premixed superheated propane and air propellants shows that the shock type oscillations predominated when the latter occurred in motors longer than 16.3 in. (see Table 1). In motors having combustion chambers shorter than 12.6 in., the pressure oscillations were predominantly of the sinusoidal type; and their amplitude was only 4 psi, measured from peak to peak, compared to 22 psi for the shock type occurring in the longer motors. Finally, with the shortest combustion chamber (7 in. long) no longitudinal combustion pressure oscillations of either type were observed.

The foregoing results indicate that the length of the combustion chamber has a decisive influence upon the magnitude and type of the combustion pressure oscillations of the longitudinal mode. Moreover, from the observations one can postulate a mechanism for explaining the influence of the length of the combustion chamber upon the magnitude of the combustion pressure oscillations of a longitudinal mode.

Consider a small pressure disturbance traveling longitudinally downstream from the injector toward the nozzle. During the time that the pressure disturbance is moving in the downstream direction, the combustion process in the vicinity of the injector proceeds at a normal rate. After the pressure disturbance has been reflected upstream from the nozzle and just prior to its arrival at the injector face, it must traverse recently ignited and unburned propellants. The pressure disturbance compresses the unburned and burning mixture thereby increasing its temperature and consequently its burning rate. The increased burning rate, in turn, amplifies the pressure disturbance, because additional hot combustion gases are evolved at a rate faster than that for the steady-state combustion pressure. The amplified pressure disturbance is reflected from the injector face and again passes through the unburned mixture further increasing its burning rate and thereby further amplifying the pressure disturbance.

It may, therefore, be expected that in the case of rocket motors having a short tubular combustion chamber, so that the time for the pressure disturbance to travel twice the length of the combustion chamber is short, there will be only a small quantity of combustible mixture injected in the time interval for the reflections of two successive pressure waves from the face of the injector. Because of the smaller quantities of unburned and burning propellants traversed by the pressure disturbance the smaller will be the energy release, and consequently the combustion pressure oscillations, if they occur, will be of small amplitude. It is also conceivable that if the combustion chamber is short enough the amplitude of the combustion pressure oscillations of a longitudinal mode may be too small to be significant. The length of combustion chamber corresponding to the latter condition has been termed the *critical combustion chamber length.*

A long combustion chamber affords the pressure disturbance a relatively long time for traversing twice the length of the combustion chamber, and likewise increases the amounts of unburned and burning propellants between successive reflections of the pressure wave. Consequently, a large pressure amplification may be expected because of the larger amount of thermochemical energy released by the compression of the large quantities of unburned and burning propellants by the pressure disturbance.

From Table 2 it is apparent that the critical length of the combustion chamber varies with the propellant combination. There is reason for assuming that the critical length of the combustion chamber is related to the flame speed of the propellant combination (the ethylene-air combination has the highest flame speed and methane-air the lowest for the propellants investigated) (3).

Experimenters using liquid bipropellant rocket motors have also noted that long combustion chambers tended to initiate combustion pressure oscillations of the longitudinal mode more readily than do short ones (4, 5). The mechanism postulated in the foregoing, while based on experiments with rocket motors burning premixed gases, also explains the greater tendency of rocket motors burning liquids to be more susceptible to high-frequency combustion pressure oscillations when the combustion chamber is long. The same mechanism can be employed for explaining why large liquid propellant rocket motors, since they have small aspect ratios, are apt to excite the transverse and tangential modes of combustion pressure oscillation (6, 7).

The Effect of Nozzle Shape Study

The results of the study of the influence of the shape of the nozzle (see Figs. 7 and 8) appear to confirm the mechanism of reverberating shock waves described in the foregoing as an explanation of the mechanism for sustaining the longitudinal mode of combustion pressure oscillation. Fig. 8 indicates that when the converging portion of the nozzle does not furnish a surface from which pressure waves can be reflected with their full intensity, the wave shape of the longitudinal mode of combustion pressure oscillation will not be the shock type.

Steady-State Combustion Pressure and Equivalence Ratio

The data indicate that increasing the steady-state combustion pressure from 52 to 130 psia caused the peak to peak amplitude of the shock type of oscillation to increase from 20 to 35 psi. A larger amplitude is to be expected at the higher steady-state pressure because of the greater mass of unburned and slow-burning propellants that become available for compression by a pressure wave in a combustion chamber of a given volume. The resulting shock wave, therefore, released a greater amount of energy thereby feeding back more pressure energy into the wave.

For all of the premixed gaseous propellant combinations

employed, but more pronounced for the ethane-air and the hydrogen-air combinations, it was noted that increasing the steady-state combustion pressure increased the amplitude of a transverse mode. If it is postulated that the mechanism by which the amplitude of a transverse mode is increased should be similar to that by which the amplitude of the longitudinal mode is increased, the transverse mode must also consist of shock fronted pressure waves. Future experimental work with larger diameter motors has been planned for a thorough investigation of the factors influencing the transverse modes.

Conclusions

The mechanism which initiates the high-frequency combustion pressure oscillations observed in the experiments is not completely understood. It appears quite likely that the combustion process introduces momentary pressure gradients which are capable of initiating small pressure disturbances, and the latter initiate the combustion pressure oscillations if certain conditions, all of which have as yet not been investigated, are favorable for their amplification. Factors having a bearing on the character of the oscillations that have been investigated at Purdue University are the combustion chamber length, nozzle shape, equivalence ratio, combustion pressure, and propellant combination.

It may be concluded from the subject experiments that the shock fronted pressure waves are a type of detonation wave traveling longitudinally in the combustion chamber. The wave is sustained by the extremely rapid release of chemical energy immediately behind the shock front. Thus, the shock front and the energy release are interrelated in that the shock front causes the rapid energy release, and the rapid energy release serves as the driving force for maintaining the shock wave.

The sinusoidal shaped waves discussed in the foregoing appear to be a type of low-amplitude pressure wave in which the energy release occurs at a relatively slow rate. It would seem reasonable to assume that the rate of energy release is related to the thermochemical properties of the unburned premixed gaseous propellants, the dimensions of the combustion chamber, and the pressure change effected by the traveling pressure front.

Experiments are either in progress or planned at the Jet Propulsion Center, Purdue University, in which the effects of changes in the combustion chamber diameter and other combustion chamber conditions are to be investigated.

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Boundary Layers With Chemical Reactions Due to Mass Addition

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This analysis is concerned with the essential features of the chemical interaction between a gaseous boundary layer and sublimed or ablated material from a body surface or gas injected through a porous wall. These effects are in addition to the "blowing" effect of the gas added to the boundary layer. The models studied are those for which the diffusion coefficients of all species are equal, the "frozen" Prandtl and Lewis moduli are taken as unity, and reactions proceed at an infinite rate. Solutions are given for three problems: (a) A diffusion-limited irreversible reaction, typified by the oxidation of carbon to carbon monoxide, (b) a temperature-limited irreversible reaction, such as dissociation without recombination and (c) a reversible reaction including complete dissociation and recombination within the boundary layer.

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Nomenclature

B' = modified blowing parameter, $2(\rho v)_w/c_f\rho_e u_e$

c_f = skin friction coefficient, $2\tau_w/\rho_e u_e^2$

c_h = heat transfer coefficient, $\dfrac{\mu_w(\partial H/\partial y)_w}{\rho_e u_e(H_e - H_w)}$

C_i = concentration by weight of species i, ρ_i/ρ

c_p = specific heat, $\partial h/\partial T$

$\bar{c}_p$ = frozen specific heat, $\sum_i C_i c_{p_i} = \frac{\partial h}{\partial T} - \sum_i h_i \frac{\partial C_i}{\partial T}$
c_{p_i} = specific heat of a single species
D = mass diffusion coefficient
E = internal energy
h = enthalpy of mixture, $\sum_i C_i h_i$
h_i = enthalpy of species i, $E_i + (p/\rho)_i$
H = total enthalpy of mixture, $\sum_i C_i h_i + (u^2/2)$
k = thermal conductivity
L = latent heat of sublimation
Le_i = Lewis modulus, $\rho D_i \bar{c}_p / k$
m_i = mass source of species i per unit area
M = product of molecular weight and stoichiometric molar reaction coefficient
n = stoichiometric molar reaction coefficient
p = pressure
P = product species of a chemical reaction
$\overline{Pr}$ = Prandtl modulus, $\mu \bar{c}_p / k$
q = heat transfer rate per unit area
R = reactant of a chemical reaction
Sc_i = Schmidt modulus, $\mu / \rho D_i$
S = sublimate or injected gas
T = temperature
u = velocity in x-direction
$\bar{u}$ = velocity ratio, u/u_e
v = velocity in y-direction
w = mass source rate per unit volume
$x, \bar{x}$ = longitudinal coordinate along surface
y = coordinate normal to surface
λ = fraction of sublimate or injected gas entering into reaction
μ = dynamic viscosity
ρ = mass density
τ = shear stress, $\mu(\partial u/\partial y)$
$\bar{\tau}$ = shear ratio, τ/τ_w

Subscripts

C = carbon
CO = carbon monoxide
crit = critical
e = boundary layer outer edge, "external"
g = gas phase
i = species identification index
j = source (or sink) location index
O = oxygen
P = product
R = reactant
ref = evaluated at reference conditions
S = sublimate or injected gas
w = evaluated at wall
$*$ = at the reaction plane, in the case of a single reaction plane
1,2 = refers to reaction planes

Introduction

THE recent advances in boundary layer theory which include effects of dissociation, ionization, recombination and other chemical phenomena have almost entirely been restricted to the chemistry of air. (An excellent review of such treatments is given by Fay and Riddell (1).)[4] However, with increasing interest in systems utilizing melting or subliming surfaces or injected gases, it is necessary to consider the fact that significant quantities of surface or injected material may enter the boundary layer and react with it. The case of surface reactions has been treated by Emmons (2) and others.

The purpose of this paper is to present in a brief manner the essential effects of chemical reactions involving sublimated or injected gases within the boundary layer upon the amount of heat transferred to the ablating or porous surface. In this treatment, the diffusion coefficients of all species are assumed equal and the "frozen" Prandtl and Lewis moduli are taken to be unity. This allows considerable mathematical simplification and yet it is believed that the essential features of the phenomenon are retained, as Lees (3) and others have demonstrated for nonreacting boundary layers. The present paper is restricted to the case of laminar flow so that the number of assumptions may be kept to a minimum and a mathematical analysis may be carried out. Since there is no apparent reason why the chemical reactions treated herein introduce new differences between laminar and turbulent flow, it is likely that the utility of the enthalpy-velocity and similar relations can be extended to turbulent flows with the same degree of effectiveness as has been done for nonreacting flows.

In treating the chemistry of reacting boundary layers, one can attempt to use realistic approximations for reaction rates. This requires either exceedingly complicated mathematical procedures, or subsequent limiting postulates. For example, Dooley (4) assumed the concentrations of species at the surface to be known and obtained some very informative results. Another procedure is to assume infinite reaction rates (sharp flame fronts) so that concentration need not be arbitrarily specified and to proceed on that basis. Such is the treatment of this paper.

Before one can treat a particular example to determine the heat input (and rate of ablation, if such is the case) it is necessary to know the skin friction coefficient c_f (and consequently the Stanton number c_h) which is applicable to that case. This wall shear is strongly affected by the blowing action at the wall in addition to the effects of chemical reaction treated herein. The former influence, however, has been well discussed in the literature ((5) for example) and so will not be treated. In this paper the emphasis will be on the new aspects introduced by the presence of chemically active species.

For certain problems, where species concentrations are known at the wall and at the boundary layer edge, a more concise approach may be used to obtain the heat transfer to the surface; this is done, for example, by Bromberg and Lipkis in (6). The method of the present paper allows more complete knowledge of the state of the boundary layer to be obtained.

Analysis

The applicable equations for a steady laminar compressible boundary layer are

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad \ldots\ldots [1]$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} \quad \ldots\ldots [2]$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + \sum_i \rho D_i h_i \frac{\partial C_i}{\partial y} + \mu \frac{\partial}{\partial y} \frac{u^2}{2}\right) \quad [3]$$

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\rho D_i \frac{\partial C_i}{\partial y}\right) + w_i \quad \ldots\ldots [4]$$

In Equations [3, 4] it has been postulated that the diffusion of mass due to thermal and pressure gradients is negligible. Gaseous radiation has not been considered in this analysis.

The energy equation presented above does not include a term for "extraneous" energy sources, such as electrical dissipation, since such processes are not considered here. However, it should be noted that the internal energy, E_i, of each species does include *all* forms of energy which that species can potentially release by any process. Therefore, it includes the energy ("heat of formation") which is released in chemical reactions (species transformations) evaluated at some (arbitrary) reference condition. Although it may not be immediately apparent that the heat of reaction is contained implicitly in the energy equation, it can be made to appear explicitly by introducing a specific reference enthalpy. Then the differ-

[4] Numbers in parentheses indicate References at end of paper.

entiation of $\sum_i C_i h_i$ yields terms which can be combined with the appropriate mass rate of formation w_i, from Equation [4], to yield a term $\sum_i w_i h_{i,\,ref}$ which is the heat of reaction at the reference condition (6). For the present paper, the energy equation in the form of Equation [3] is employed.

Returning to Equations [1 through 4], it proves useful to identify therein the Prandtl, Lewis and Schmidt moduli

$$\overline{Pr} = \frac{\mu \bar{c}_p}{k} \qquad Le_i = \frac{\rho D_i \bar{c}_p}{k} \qquad Sc_i = \frac{\mu}{\rho D_i}$$

Then Equations [3, 4] become

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\mu}{\overline{Pr}}\left(\frac{\partial H}{\partial y}\right) + \mu\left(1 - \frac{1}{\overline{Pr}}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \sum_i \rho D_i\left(1 - \frac{1}{Le_i}\right) h_i \frac{\partial C_i}{\partial y}\right] \ldots [5]$$

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Sc_i}\frac{\partial C_i}{\partial y}\right) + w_i \ldots\ldots [6]$$

It is apparent that considerable mathematical simplification results from postulating $\overline{Pr} = 1$ and $Le_i = 1$. Lees (3) and others have also shown that these are not unreasonable physical approximations, at least for nonreacting boundary layers. In the presence of chemical reactions, the effects of departures from these postulates may be significant, but have not been considered in the present analysis. Recalling that

$$Sc_i = \frac{\overline{Pr}}{Le_i}$$

Equations [5, 6] become

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial H}{\partial y}\right) \ldots\ldots [7]$$

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial C_i}{\partial y}\right) + w_i \ldots\ldots [8]$$

Then, following Crocco, the independent variables may be changed from (x, y) to $(\bar{x}, u)$. In that event Equations [2, 7, 8] become

$$\rho u \left(\frac{\partial y}{\partial \bar{x}}\right) - \rho v = \frac{\mu}{\tau}\left(\frac{\partial p}{\partial \bar{x}}\right) - \frac{\partial \tau}{\partial u} \ldots\ldots [9]$$

$$\rho u\left[\frac{\partial H}{\partial \bar{x}} - \frac{\tau}{\mu}\left(\frac{\partial y}{\partial \bar{x}}\right)\frac{\partial H}{\partial u}\right] + \rho v \frac{\tau}{\mu}\left(\frac{\partial H}{\partial u}\right) = \frac{\tau}{\mu}\frac{\partial}{\partial u}\left(\tau \frac{\partial H}{\partial u}\right) \ldots [10]$$

$$\rho u\left[\frac{\partial C_i}{\partial \bar{x}} - \frac{\tau}{\mu}\left(\frac{\partial y}{\partial \bar{x}}\right)\frac{\partial C_i}{\partial u}\right] + \rho v \frac{\tau}{\mu}\left(\frac{\partial C_i}{\partial u}\right) = \frac{\tau}{\mu}\frac{\partial}{\partial u}\left(\tau \frac{\partial C_i}{\partial u}\right) + w_i \ldots [11]$$

If Equation [9] is now substituted into [10 and 11], the system becomes

$$\rho\mu u \frac{\partial H}{\partial \bar{x}} - \mu \frac{\partial H}{\partial u}\frac{\partial p}{\partial \bar{x}} = \tau^2 \frac{\partial^2 H}{\partial u^2} \ldots\ldots [12]$$

$$\rho\mu u \frac{\partial C_i}{\partial \bar{x}} - \mu \frac{\partial C_i}{\partial u}\frac{\partial p}{\partial \bar{x}} = \tau^2 \frac{\partial^2 C_i}{\partial u^2} + \mu w_i \ldots\ldots [13]$$

These equations can be further simplified by postulating H and C_i to depend upon velocity only; that is, $\partial H/\partial \bar{x} = \partial C_i/\partial \bar{x} = 0$. This postulate is often given the name "similarity." It may be shown to be self-consistent although the question of uniqueness has not been demonstrated. If, in addition, one considers the so-called "flat-plate" problem for which the pressure is taken as a constant, there results

$$\tau^2 \frac{\partial^2 H}{\partial u^2} = 0 \ldots\ldots [14]$$

$$\tau^2 \frac{\partial^2 C_i}{\partial u^2} + \mu w_i = 0 \ldots\ldots [15]$$

Two facts are immediately apparent from this system:

1 The total enthalpy profile is linear with velocity, independent of whether or not chemical reactions occur (provided no "external" energy sources are present; e.g., electrical dissipation). This will enable the effect of reactions upon heat transfer to be simply accounted for by means of the species composition at the surface.

2 The concentration of a given species is linear in velocity except in reaction zones.[5]

Equation [15] may be integrated twice with respect to u to get

$$C_i - C_{i,\,w} = \left(\frac{\partial C_i}{\partial u}\right)_w u - \int_0^u du'' \int_0^{u''} \frac{\mu w_i}{\tau^2}\, du' \ldots [16]$$

where the primes indicate dummy variables. If $(\partial C_i/\partial u)_w$ is determined by evaluating Equation [16] at the outer edge, $u = u_e$, and is resubstituted into [16]

$$C_i - C_{i,\,w} = \frac{u}{u_e}\left[(C_{i,e} - C_{i,w}) + \int_0^{u_e} du'' \int_0^{u''} \frac{\mu w_i}{\tau^2}\, du'\right] - \int_0^u du'' \int_0^{u''} \frac{\mu w_i}{\tau^2}\, du' \ldots [17]$$

The inside integral of the double quadrature can be written

$$\int_0^u \frac{\mu w_i}{\tau^2}\, du = \int_0^u \frac{\mu w_i du}{\tau\left(\mu \dfrac{\partial u}{\partial y}\right)} = \int_0^y \frac{w_i dy}{\tau} \ldots\ldots [18]$$

The latter form suggests an approximation for the reaction zone to be a sharp thin flame front, or an ensemble of such fronts. In that event, Equation [18] becomes

$$\int_0^u \frac{\mu w_i}{\tau^2}\, du = \sum_{u=0}^{u} \frac{m_{ij}}{\tau_j} \ldots\ldots [19]$$

where m_{ij} is the mass source rate of species i per unit area at the jth position between the surface and the coordinate position u. It should be noted that $m = 0$ *everywhere but at these reaction planes* so that the shear stress need only be associated

[5] Implicit in this statement and in the development which follows is that the quantity $(\mu w_i/\tau^2)$ is a function of u only, a condition which is assured by the postulate of infinite reaction rates. Only for this model have the conditions for separation of variables been considered; the general conditions for separability have not been examined.

with these locations. The u at the upper limit of the summation indicates that only sources between the surface and the position where the velocity has the value u are to be considered.

With this notation convention, the double integral can be written

$$\int_0^u du'' \int_0^{u''} \frac{\mu w_i du'}{\tau^2} = u \sum_0^u \frac{m_{ij}}{\tau_j} - \sum_0^u \frac{m_{ij} u_j}{\tau_j} \quad \ldots [20]$$

and Equation [16] becomes

$$C_i - C_{i,w} = \left(\frac{\partial C_i}{\partial u}\right)_w u - \sum_0^u \frac{m_{ij}}{\tau_j}(u - u_j) \quad \ldots [21]$$

and Equation [17] becomes

$$C_i - C_{i,w} = \frac{u}{u_e}\left[(C_{i,e} - C_{i,w}) + \sum_u^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j)\right] + \left(1 - \frac{u}{u_e}\right) \sum_0^u \frac{m_{ij} u_j}{\tau_j} \quad \ldots [22]$$

Heat Transfer

The concentration relations defined in the preceding section are now applied to determine the resulting effect on heat transfer. Since

$$\left(\frac{\partial H}{\partial y}\right)_{g,w} = \left[\frac{\partial}{\partial y}\left(\sum_i C_i h_i + \frac{u^2}{2}\right)\right]_{g,w}$$

$$= \left[\frac{\partial T}{\partial y} \bar{c}_p + \sum_i h_i \frac{\partial C_i}{\partial y}\right]_{g,w}$$

one may write

$$\left(\frac{\partial T}{\partial y}\right)_{g,w} = \frac{1}{\bar{c}_p}\left[\left(\frac{\partial H}{\partial y}\right)_w - \left(\sum_i h_i \frac{\partial C_i}{\partial y}\right)_{g,w}\right]$$

or, for $\overline{Pr} = 1$

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} = \left(\mu \frac{\partial H}{\partial y}\right)_{g,w} - \left(\mu \frac{\partial u}{\partial y}\right)_w \left(\sum_i h_i \frac{\partial C_i}{\partial u}\right)_{g,w} = \left(\mu \frac{\partial H}{\partial y}\right)_{g,w} - \frac{\tau_w}{u_e}\left(\sum_i h_i \frac{\partial C_i}{\partial \bar{u}}\right)_{g,w} \quad \ldots [23]$$

For a subliming wall recall that[6]

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} = \left(k \frac{\partial T}{\partial y}\right)_{\text{solid},w} + L(\rho v)_w \quad \ldots [24]$$

The concentration gradients needed for Equation [23] can be obtained from Equation [21 or 22]

$$\left(\frac{\partial C_i}{\partial \bar{u}}\right)_w = C_{i,e} - C_{i,w} + \sum_0^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j) \quad \ldots [25]$$

yielding, for the heat transfer at the wall

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} = \left(\mu \frac{\partial H}{\partial y}\right)_{g,w} - \frac{\tau_w}{u_e}\left\{\sum_i h_{i,w}(C_{i,e} - C_{i,w}) + \sum_i \left[h_{i,w} \sum_0^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j)\right]\right\} \quad \ldots [26]$$

where $\sum_i$ is the summation over all species and $\sum_0^{u_e}$ is the summation over all reaction planes in the interval 0 to u_e. The first term on the right, expressed in the usual form, is $c_h \rho_e u_e (H_e - H_w)$ or, more precisely[7]

$$\left(\mu \frac{\partial H}{\partial y}\right)_{g,w} = c_h \rho_e u_e \left\{\sum_i [(C_i h_i)_e - (C_i h_i)_w] + \frac{u_e^2}{2}\right\}$$

In the second term on the right of [26], τ_w may be replaced by $(c_f/2)\rho_e u_e^2$, which for $\overline{Pr} = 1$ equals $c_h \rho_e u_e^2$. On introducing these relations, Equation [26] becomes[8]

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} = c_h \rho_e u_e \left\{\left[\sum_i C_{i,e}(h_{i,e} - h_{i,w}) + \frac{u_e^2}{2}\right] - \sum_i h_{i,w} \sum_0^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j)\right\} \quad \ldots [27]$$

The term in square brackets deserves some particular attention. It is *not* the usual difference between external and wall enthalpy, but is the enthalpy difference from the stream to the surface conditions, *of the external species*, regardless of whether they actually exist at the surface. The second term is the reaction contribution, which will be shown to be equal to the heat of reaction appropriate to the particular example.

The mass of a species passing any plane parallel to the wall is

$$m_i = C_i \rho v - \rho D_i \frac{\partial C_i}{\partial y} = C_i \rho v - \tau \frac{\partial C_i}{\partial u} \quad \ldots [28]$$

Inasmuch as concentration is a linear function of u, the above expression for m_i may be rewritten in terms of species concentrations at known locations. It is therefore of interest to determine these concentrations.

The general solution for the concentration of a species for

[6] This equation is correct if there are no chemical reactions at the wall. In the general case which may include surface reactions

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} - \sum_i (\rho_i v_i h_i)_{g,w} = \left(k \frac{\partial T}{\partial y}\right)_{\text{solid},w} - (\rho v h)_{\text{solid},w}$$

so that an additional term would appear on the right of Equation [24], namely

$$\left[\sum_i (\rho_i v_i h_i)_{g,w} - (\rho v)_{\text{solid},w} h_{S,g,w}\right]$$

See also footnote pertaining to Equation [27]. Radiation from the wall, omitted here, could be included by adding the usual $\epsilon \sigma T^4$ term on the right of Equation [24].

[7] Note that c_h is defined here in terms of the actual species present at the boundary layer edge and at the wall.

[8] In cases where reactions occur on the surface, the surface reaction terms are not to be included in the double summation of Equation [27], which applies to gas phase reactions only. The surface reaction terms are already accounted for, as noted in the footnote to Equation [24]. Another intuitively satisfying approach, which yields the same result, is to conceive of the surface reaction as a gas phase reaction just off the surface (where $u = 0$) and to treat it in the general manner described herein.

the condition of several planes of infinitely fast reactions occurring in the boundary layer is given in Equation [22]. The diffusion boundary condition for the sublimate or injected gas is

$$(1 - C_{S,w})(\rho v)_w = -\rho_w D_S \left(\frac{\partial C_S}{\partial y}\right)_w = -\mu_w \left(\frac{\partial C_S}{\partial y}\right)_w \ldots [29]$$

and, for each of the other species, is

$$0 = C_{i,w}(\rho v)_w - \rho D_i \left(\frac{\partial C_i}{\partial y}\right)_w = C_{i,w}(\rho v)_w - \mu_w \left(\frac{\partial C_i}{\partial y}\right)_w \ldots [30]$$

These boundary conditions arise from considering that, for the sublimate or injected gas, the mass rate of the species is the total mass rate at the surface, and, for all other species, the convection outward is just balanced by the diffusion inward to yield zero mass flow at the wall ((5), for example). From the general solution we have

$$\left(\frac{\partial C_i}{\partial y}\right)_w = \frac{1}{u_e}\left(\frac{\partial u}{\partial y}\right)_w \left[C_{i,e} - C_{i,w} + \sum_0^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j)\right] = \frac{1}{\mu_w}\frac{c_f}{2}\rho_e u_e \left[C_{i,e} - C_{i,w} + \sum_0^{u_e} \frac{m_{ij}}{\tau_j}(u_e - u_j)\right] \ldots [31]$$

Combining [29 and 31] yields the concentration at the wall for the sublimate or injected gas

$$C_{S,w} = \frac{B'}{B'+1}\left[1 - \sum_0^{u_e} \lambda_j \left(\frac{1 - \bar{u}_j}{\bar{\tau}_j}\right)\right] \ldots [32]$$

where B' is defined in terms of the actual skin friction coefficient rather than the more usual no-blowing skin friction coefficient; λ_j is the fraction of the sublimate produced at the wall which is consumed at each reaction plane j, that is

$$\lambda_j \equiv \frac{-m_{S,j}}{(\rho v)_w}$$

The wall concentration of all other species is found from Equations [30, 31]

$$C_{i,w} = \frac{1}{B'+1}\left[C_{i,e} - \frac{M_i}{M_S} B' \sum_0^{u_e} \lambda_j \left(\frac{1 - \bar{u}_j}{\bar{\tau}_j}\right)\right] \ldots [33]$$

where M_i is the product of molecular weight and stoichiometric molar reaction coefficient of species i.

For a single plane reaction represented by

$$n_S S + n_R R \rightarrow n_P P \ldots [34]$$

the concentrations of the sublimate, reactant and product of the reaction and of any inert gas present, are

$$C_{S,w} = \frac{B'}{B'+1}\left(1 - \lambda_* \frac{1 - \bar{u}_*}{\bar{\tau}_*}\right) \ldots [35]$$

$$C_{R,w} = \frac{1}{B'+1}\left[C_{R,e} - \frac{M_R}{M_S}\lambda_* B' \left(\frac{1 - \bar{u}_*}{\bar{\tau}_*}\right)\right] \ldots [36]$$

$$C_{P,w} = \frac{B'}{B'+1}\frac{M_P}{M_S}\lambda_* \left(\frac{1 - \bar{u}_*}{\bar{\tau}_*}\right) \ldots [37]$$

$$C_{\text{inert},w} = \frac{C_{\text{inert},e}}{B'+1} \ldots [38]$$

Since in the case of a single reaction plane all of the sublimate or injected gas which reaches that plane is consumed, an additional expression for $C_{S,w}$ can be obtained by setting $C_{S,*} = 0$ in Equation [22] and solving for $C_{S,w}$. This yields

$$C_{S,w} = \frac{\lambda_* B' \bar{u}_*}{\bar{\tau}_*} \ldots [39]$$

Equating [35 and 39] yields an expression for the fraction of the sublimed or injected material which reaches the (single) reaction plane

$$\lambda_* = \frac{\bar{\tau}_*}{1 + B'\bar{u}_*} \ldots [40]$$

The concentration of the reactant at the reaction plane is obtained from [22 and 36]. For a single reaction plane, on combining with [40]

$$C_{R,*} = \left(\frac{1 + B'\bar{u}_*}{1 + B'}\right) C_{R,e} - \frac{B'}{B'+1}\frac{M_R}{M_S}(1 - \bar{u}_*) \ldots [41]$$

Similarly, the concentration of the product at the reaction plane, from [22, 37], is

$$C_{P,*} = \frac{B'}{B'+1}\frac{M_P}{M_S}(1 - \bar{u}_*) \ldots [42]$$

It should be noted that in obtaining the expressions for the concentration of the reaction product ([37, 42]), the concentration of the product at the outer edge of the boundary layer has been taken to be zero. This is not true in those cases where the product exists in the outer stream (such as oxygen or nitrogen), but even in these cases it is permissible to "tag" the molecules which are products of the reaction, whose concentration does go to zero at the outer edge, and to consider the remaining outer stream molecules of the same species as inerts which do not enter the reaction. The expressions above can, therefore, be used even for products which also exist in the outer stream.

It is now possible to apply these relations to certain idealized reactions.

Example 1. Diffusion-Limited Irreversible Reaction

Consider a combustion reaction typified by

$$C + O \rightarrow CO \ldots [43]$$

with infinite reaction rate in the forward direction and zero rate in the reverse direction, for the case in which the idealized critical temperature for the reaction T_{cr} is below the wall temperature: $T_{crit} < T_w$. The critical temperature is that below which it is postulated that the reaction cannot occur and above which it must occur. The concentration profiles in the $\bar{u}$-coordinate can be either the profiles shown in Fig. 1 or those shown in Fig. 2.

Fig. 1 represents the case of reaction at the surface. There are two fundamentally different situations which might lead to this:

1 The wall temperature is appreciably below the "sublimation temperature," so that a solid phase reaction takes place

$$C_{solid} + O \rightarrow CO$$

2 The sublimation rate of C is less than the stoichiometric quantity compared with the rate of supply of O by diffusion, so that all the sublimed carbon is immediately consumed at the wall by the reaction.

Fig. 2 represents the case where the mass rate of carbon sublimed is greater than can be combined with all the oxygen atoms reaching the surface per unit time, so that the reaction plane moves out into the boundary layer to a point where stoichiometric ratios exist and both C and O concentrations go to zero.

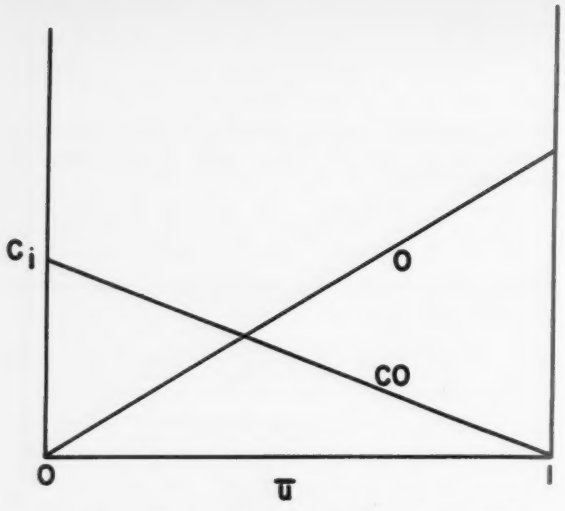

Fig. 1 Concentration distribution for example 1 (surface combustion)

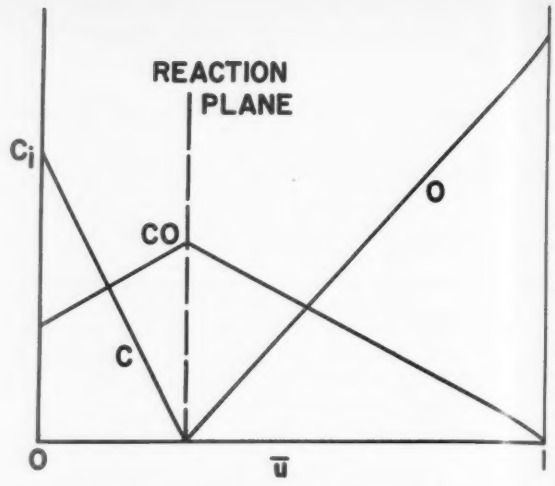

Fig. 2 Concentration distribution for example 1 (gas-phase combustion)

From [28] the mass of O consumed at the reaction plane is

$$m_{O,*} = C_{O,*}\,\rho v - \tau_* \left(\frac{C_{O,e} - C_{O,*}}{u_e - u_*}\right)$$

$$m_{O,*} = -\left(\frac{\tau_*}{u_e - u_*}\right) C_{O,e}$$

since the concentration of O at the reaction plane is zero in this example. In addition

$$m_{C,*} = m_{O,*}\,\frac{M_C}{M_O} = \frac{12}{16}\,m_{O,*}$$

$$m_{CO,*} = -m_{O,*}\,\frac{M_{CO}}{M_O} = -\frac{28}{16}\,m_{O,*}$$

For a single plane reaction, the second summation of [27] becomes

$$-\sum_i h_{i,w}\,\frac{m_{i,*}}{\tau_*}\,(u_e - u_*)$$

and, for the present example, [27] becomes[9]

$$\left(k\,\frac{\partial T}{\partial y}\right)_{g,w} = c_h \rho_e u_e \left[\sum_i C_{i,e}\,(h_{i,e} - h_{i,w}) + \frac{u_e^2}{2} + C_{O,e}\left(h_{O,w} + \frac{12}{16}\,h_{C,w} - \frac{28}{16}\,h_{CO,w}\right)\right] \left(k\,\frac{\partial T}{\partial y}\right)_{g,w} =$$

$$c_h \rho_e u_e \left[\sum_i C_{i,e}\,(h_{i,e} - h_{i,w}) + \frac{u_e^2}{2} + C_{O,e}\,\Delta H_{\text{reaction},w,O}\right] \quad \ldots [44]$$

Thus, the heat transfer potential to the surface is the usual potential of the outer species plus the heat of formation of CO (calculated at the wall temperature) per pound of air. The solution is independent of the location of the reaction plane, whether at the surface or within the boundary layer, and is independent of the concentration at the wall of any species involved in the reaction. It is apparent that this same result would apply if the reaction occurred over a reaction zone of finite width made up of many reaction planes rather than on the single plane treated here.

It should be noted that, in view of the expression for $[k\,(\partial T/\partial y)]_{g,w}$ (Equation [24]), the heat of formation applicable to Equation [44] is the gas phase heat of formation, regardless of whether the reaction occurs in the gas phase or on the surface; for the reaction considered, Equation [43], this is 29,000 Btu per pound of oxygen. Further, the value of the heat transfer potential is the same whether the reaction is $C + O \rightarrow CO$ or $C + \frac{1}{2}\,O_2 \rightarrow CO$ because, while the $C + O$ reaction is more energetic, by an amount equal to the heat of dissociation of O_2, the enthalpy difference across the boundary layer for oxygen (the summation term applicable to oxygen in Equation [44]) is reduced by that same quantity for the part of the oxygen which enters the boundary layer dissociated.

Using the concentration relations previously derived, it is possible to calculate the blowing rate necessary to cause the reaction to occur within the boundary layer rather than on the wall. The critical blowing rate, below which the reaction of this example will be on the wall and above which the reaction will occur in the boundary layer, is obtained from [36 or 41]. It is

$$B'_{\text{crit}} = \frac{M_S}{M_R}\,C_{R,e} \quad \ldots [45]$$

which, for the carbon combustion described by Equation [43], is

$$B'_{\text{crit}} = 0.17$$

By setting $C_{R,*} = 0$ in Equation [41], the blowing rate necessary to push the reaction plane out to any position in the boundary layer is, in terms of the coordinate $\bar{u}$

$$B' = \frac{C_{R,e}}{(1 - \bar{u}_*)\,\dfrac{M_R}{M_S} - \bar{u}_* C_{R,e}} = \frac{B'_{\text{crit}}}{1 - \bar{u}_*\,(1 + B'_{\text{crit}})} \quad \ldots [46]$$

[9] A similar result has recently been obtained by M. R. Denison and D. A. Dooley of Aeronutronic Systems, Inc.

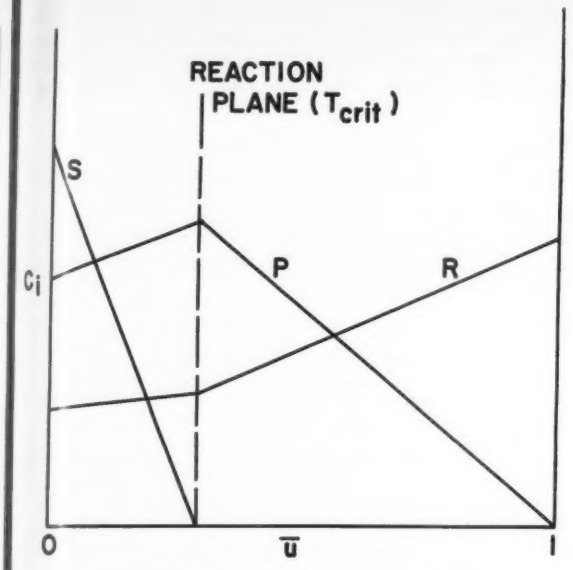

Fig. 3 Concentration distribution for example 2(a) (excess of reactant at T_{crit})

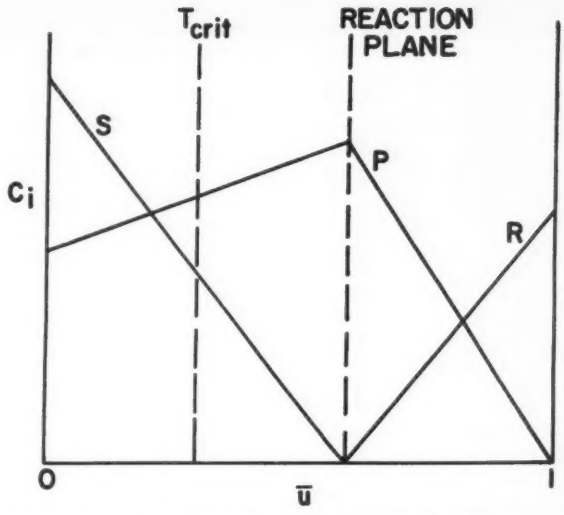

Fig. 4 Concentration distribution for example 2(a) (excess of sublimate at T_{crit})

Equation [46] indicates that the condition

$$\bar{u}_* = \frac{1}{1 + B'_{crit}} < 1$$

corresponds to $B' = \infty$ (or $c_f = 0$). Since the principal postulate of boundary layer theory, $\partial/\partial y \gg \partial/\partial x$, becomes questionable under the circumstance $(\partial u/\partial y)_w \to 0$, the present analyses must be limited to reaction somewhat inside of the plane where

$$\frac{u}{u_e} = \frac{1}{1 + B'_{crit}}$$

Example 2. Temperature-Limited Irreversible Reaction

In example 1 the idealized critical reaction temperature was below the wall temperature so that the reaction would occur whenever an S molecule encountered an R molecule. The whole process was limited only by diffusion and convection activity. In the present example the critical temperature for reaction is above the wall temperature so that the reaction cannot occur in the region between the wall and the idealized critical temperature profile. In other respects the postulates of the problem are unchanged: There is a single reaction plane; the forward reaction rate is infinite, and the reverse rate is zero. Two somewhat similar cases are considered: Chemical combination of the sublimate or injected gas with the reactant from the outer stream, or dissociation of the sublimate involving no chemical reaction with the outer stream.

(a) *Chemical combination of sublimate with reactant.* The concentration profiles in the $\bar{u}$-coordinate are shown in Figs. 3 and 4. As before, the sublimate disappears at the reaction plane. The reaction product goes to zero at the edge of the boundary layer (by definition of the edge) but is finite at the wall. The reactant from the outer stream (usually oxygen) will still be present at the wall if there is an excess of it at the reaction plane (Fig. 3). If there is an excess of sublimate at the critical reaction temperature, the plane of reaction will move out to the point where the sublimate and reactant reach stoichiometric proportions, at which point the concentrations of both will go to zero (Fig. 4).

Equation [27] still applies, but now allowance is made for the possibility that the reactant will not be used up at the reaction plane; the amount consumed will be the difference between that entering and that leaving the reaction plane

$$-m_{R,*} = [(\rho v)_R]_{u=u_*+} - [(\rho v)_R]_{u=u_*-} = \left[\rho D_R \left(\frac{\partial C_R}{\partial y}\right)\right]_{u_*+} - \left[\rho D_R \left(\frac{\partial C_R}{\partial y}\right)\right]_{u_*-}$$

For $\overline{Pr} = Le = 1$, $\rho D = \mu$. On replacing $\partial C/\partial y$ by $(\partial C/\partial u)\cdot(\partial u/\partial y)$

$$-m_{R,*} = \tau_* \left[\left(\frac{\partial C_R}{\partial u}\right)_{u_*+} - \left(\frac{\partial C_R}{\partial u}\right)_{u_*-}\right]$$

The concentration profiles are linear in u, thus

$$-m_{R,*} = \frac{\tau_*}{u_e - u_*}\left[(C_{R,e} - C_{R,*}) - \frac{u_e - u_*}{u_*}(C_{R,*} - C_{R,w})\right]$$

On substituting $C_{R,w}$ from Equations [36, 40], and for $C_{R,*}$ from [41], the mass of reactant consumed at the reaction plane becomes

$$-m_{R,*} = \left(\frac{\tau_*}{u_e - u_*}\right)\frac{\frac{M_R}{M_S}B'(1 - \bar{u}_*)}{1 + B\bar{u}_*}$$

As before

$$m_{S,*} = \frac{M_S}{M_R}m_{R,*} \text{ and } m_{P,*} = -\frac{M_P}{M_R}m_{R,*}$$

The heat transfer is

$$\left(k\frac{\partial T}{\partial y}\right)_{e,w} = c_h\rho_e u_e\left[\sum_i C_{i,e}(h_{i,e} - h_{i,w}) + \frac{u_e^2}{2} + \left(h_{R,w} + \frac{M_S}{M_R}h_{S,w} - \frac{M_P}{M_R}h_{P,w}\right)\frac{M_R B'(1 - \bar{u}_*)}{M_S(1 + B'\bar{u}_*)}\right]\left(k\frac{\partial T}{\partial y}\right)_{e,w} =$$

$$c_h\rho_e u_e\left[\sum_i C_{i,e}(h_{i,e} - h_{i,w}) + \frac{u_e^2}{2} + \Delta H_{\text{reaction},w,R}\frac{M_R B'(1 - \bar{u}_*)}{M_S(1 + B'\bar{u}_*)}\right] \quad \ldots [47]$$

where $\Delta H_{\text{reaction, } w, R}$ is the heat of reaction per pound of R evaluated at wall conditions. This equation requires knowledge of the location of the reaction $\bar{u}_*$. An alternative form requiring, instead, knowledge of the wall concentration of the reactant can be obtained by employing Equations [36, 40]

$$\left(k\frac{\partial T}{\partial y}\right)_{g,\,w} = c_h\rho_e u_e\left\{\sum_i C_{i,\,e}(h_{i,\,e}-h_{i,\,w}) + \frac{u_e^2}{2} + \Delta H_{\text{reaction } w,\,R}[C_{R,\,e} - (B'+1)C_{R,\,w}]\right\} \quad [48]$$

This is similar to the diffusion limited case of example 1 except that here the heat of reaction is modified by a quantity which is a function of the position of the reaction plane in the boundary layer, rather than the constant quantity $C_{R,\,e}$ of example 1. If the critical reaction temperature is reduced so that the reaction moves to the wall, the solution should agree with that of example 1. In Equation [47] as the reaction moves to the wall, $\bar{u}_* \to 0$ and the multiplier of the heat of reaction becomes

$$\frac{M_R}{M_S}B'$$

Equation [41] shows that for the conditions of example 1, with the reaction on the wall ($C_{R,\,*} = 0$ and $\bar{u}_* = 0$)

$$\frac{M_R}{M_S}B' = C_{R,\,e}$$

Thus example 2 contains example 1 as a special case.

When the condition of Fig. 4 is achieved, the temperature limitation is removed and diffusional processes govern. The solution should then agree with that of example 1 for the condition where the blowing rate (of sublimate or injected gas) is great enough to force the reaction off the wall. The blowing rate necessary to cause the reaction to occur to the right of the plane of critical temperature (Fig. 4) is precisely that given by Equation [46] ($C_{R,\,*} = 0$). Substituting this in Equation [47] leads to

$$\left(k\frac{\partial T}{\partial y}\right)_{g,\,w} = c_h\rho_e u_e\left[\sum_i C_{i,\,e}(h_{i,\,e}-h_{i,\,w}) + \frac{u_e^2}{2} + C_{R,\,e}\,\Delta H_{\text{reaction, } w,\,R}\right] \quad [49]$$

Therefore, as long as the blowing rate is such that the reaction occurs farther from the surface than the critical temperature plane, regardless of whether the critical temperature is greater or less than the wall temperature, the heat transferred to the wall will be independent of position of the reaction plane and will be as given by Equation [44 or 49].

(b) *Sublimate dissociation* or other similar changes of states involving a heat of formation. Under the idealized condition that the reaction rate of the reverse process (recombination) is zero, such dissociation may be typically represented by

$$S \to P_1 + P_2 + \ldots$$

where there may be one or more dissociation products of the sublimate or injected gas. This is identical to the case considered above except that there are no reactants, only sublimate and products. The mass of P_1 produced at the reaction plane, following the similar calculation above for the mass of reactant, is

$$m_{P_1,\,*} = \frac{\tau_*}{u_e - u_*}\frac{\frac{M_{P_1}}{M_S}B'(1-\bar{u}_*)}{1+B'\bar{u}_*}$$

from which

$$M_{S,\,*} = \frac{\tau_*}{u_e - u_*}\frac{B'(1-\bar{u}_*)}{1+B'\bar{u}_*}$$

The heat transfer, expressed in terms of the sublimate, is

$$\left(k\frac{\partial T}{\partial y}\right)_{g,\,w} = c_h\rho_e u_e\left[\sum_i C_{i,\,e}(h_{i,\,e}-h_{i,\,w}) + \frac{u_e^2}{2} + \Delta H_{\text{reaction, } w,\,S}\frac{B'(1-\bar{u}_*)}{1+B'\bar{u}_*}\right] =$$

$$c_e\rho_k u_e\left\{\sum_i C_{i,\,e}(h_{i,\,e}-h_{i,\,w}) + \frac{u_e^2}{2} + \Delta H_{\text{reaction, } w,\,S}[B' - (B'+1)C_{S,\,w}]\right\} \quad [50]$$

where $\Delta H_{\text{reaction, } w,\,S}$, the heat of reaction or dissociation per pound of sublimate evaluated at wall conditions, would normally be negative, thus reducing the heat transfer. In the case of a dissociation reaction, no reactant from the outer edge of the boundary layer is involved, so the remarks in part (a) above regarding the critical blowing rate do not apply; the dissociation will occur at the specified temperature plane no matter how great the blowing rate.

An alternate interpretation of the results of examples 1 and 2 should be pointed out. In example 1, the effect of the combustion is to add a constant term to the boundary layer enthalpy potential (Equation [44]); the combustion reaction, from the point of view of the surface, looks like an increase in that potential (for an exothermic reaction). The situation in example 2 may be viewed differently. These solutions (for example, Equations [47, 50]) can be written in the form

$$\left(k\frac{\partial T}{\partial y}\right)_{g,\,w} = c_h\rho_e u_e\left[\sum_i C_{i,\,e}(h_{i,\,e}-h_{i,\,w}) + \frac{u_e^2}{2}\right] + (\rho v)_w\left[\Delta H_{\text{reaction}}\frac{\frac{M_R}{M_S}(1-\bar{u}_*)}{\left(1+\frac{(\rho v)_w\bar{u}_*}{c_h\rho_e u_e}\right)}\right]$$

The chemical reaction is of greatest importance when $\bar{u}_*$ is small (the term approaches zero when $\bar{u}_*$ approaches unity), in which case the denominator of the bracketed quantity approaches unity. For the temperature-limited reaction, $\bar{u}_*$ is constant so the last term can be approximated as $(\rho v)_w L'$, where L' is the bracketed quantity above and is an only slightly varying quantity (defined as positive for exothermic reactions). Recall that

$$\left(k\frac{\partial T}{\partial y}\right)_{g,\,w} = (\rho v)_w L + \left(k\frac{\partial T}{\partial y}\right)_{\text{solid, } w}$$

The example 2 solutions can now be put in the form

$$(\rho v)_w L + \left(k \frac{\partial T}{\partial y}\right)_{\text{solid},\ w} = c_h \rho_e u_e \sum_i C_{i,\ e} (h_{i,\ e} - h_{i,\ w}) + \frac{u_e^2}{2} + (\rho v)\, L'$$

or

$$(\rho v)_w (L - L') + \left(k \frac{\partial T}{\partial y}\right)_{\text{solid},\ w} = c_h \rho_e u_e \sum_i C_{i,\ e} (h_{i,\ e} - h_{i,\ w}) + \frac{u_e^2}{2}$$

so that in this case the chemical reaction or dissociation now looks like an increase (for an endothermic reaction) or decrease (for an exothermic reaction) in the latent heat of the wall material.

Example 3. Reversible Dissociation Reaction

Consider a simple reversible dissociation reaction, such as might be typified by

$$2H_2O \leftrightharpoons 2H_2 + O_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots [51]$$

where, for convenience, the dissociation is postulated to occur at temperature T_{diss} within the boundary layer, such that H_2O molecules cannot exist at a higher temperature, and recombination occurs at a lower temperature T_{recom} below which the dissociation products cannot co-exist. The concentration profiles are as shown in Fig. 5.

The example remains valid if the two reaction planes are moved together so that there is a single temperature profile separating the dissociated and combined molecules.

Since all of the H_2 is consumed at the recombination plane, the values of $C_{H_2,\ w}$ and $(\partial C_{H_2}/\partial u)_w$ are zero. $C_{H_2,\ e}$ is zero because H_2 is not present in the outer flow. Therefore, the summation term for H_2 is zero. Specifically

$$\sum_0^{u_e} \frac{m_{H_2,\ j}}{\tau_j} (u_e - u_j) = \frac{m_{H_2,\ 1}}{\tau_1} (u_e - u_1) + \frac{m_{H_2,\ 2}}{\tau_2} (u_e - u_2) = 0 \quad \ldots [52]$$

The mass source and sink rates of O_2 and H_2O at each reaction plane are tied to that of H_2 by Equation [51]

$$m_{O_2,\ j} = \frac{M_{O_2}}{M_{H2}} m_{H_2,\ j} = \frac{32}{4} m_{H_2,\ j}$$

$$m_{H_2O,\ j} = -\frac{M_{H_2O}}{M_{H_2}} m_{H_2,\ j} = -\frac{36}{4} m_{H_2,\ j}$$

Therefore, it must follow that

$$\sum_0^{u_e} \frac{m_{O_2,\ j}}{\tau_j} (u_e - u_j) = \frac{m_{O_2,\ 1}}{\tau_1} (u_e - u_1) + \frac{m_{O_2,\ 2}}{\tau_2} (u_e - u_2) = 0 \ldots\ldots [53]$$

and

$$\sum_0^{u_e} \frac{m_{H_2O,\ j}}{\tau_j} (u_e - u_j) = \frac{m_{H_2O,\ 1}}{\tau_1} (u_e - u_1) + \frac{m_{H_2O,\ 2}}{\tau_2} (u_e - u_2) = 0 \ldots\ldots [54]$$

The heat transfer to the wall (Equation [27]) is

$$\left(k \frac{\partial T}{\partial y}\right)_{g,w} = c_h \rho_e u_e \left[\sum_i C_{i,\ e} (h_{i,\ e} - h_{i,\ w}) + \frac{u_e^2}{2} - \sum_i h_{i,\ w} \sum_0^{u_e} \frac{m_{ij}}{\tau_j} (u_e - u_j)\right]$$

If Equation [21] is written for the concentration of H_2 at the outer edge, it becomes

$$C_{H_2,\ e} - C_{H_2,\ w} = \left(\frac{\partial C_{H_2}}{\partial u}\right)_w u_e - \sum_0^{u_e} \frac{m_{H_2,\ j}}{\tau_j} (u_e - u_j)$$

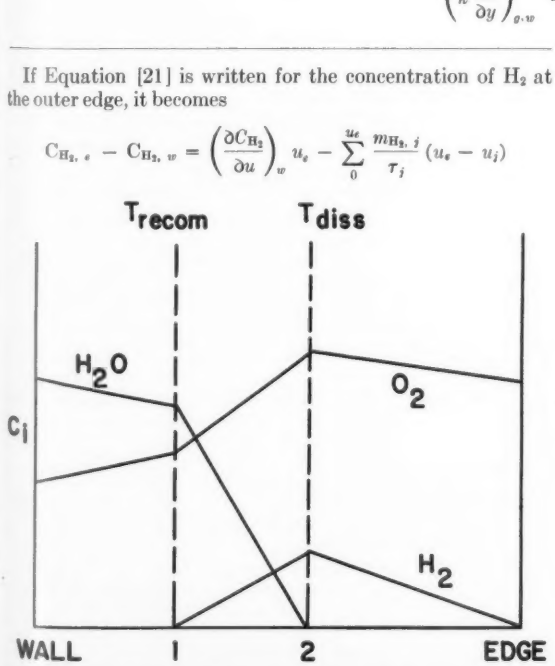

Fig. 5. Concentration distribution: Example 3, reversible dissociation reaction

Clearly, since the summation $\sum_0^{u_e} m_{ij}(u_e - u_j)/\tau_j$ of each species is separately zero (Equations [52, 53, 54]), it follows that

$$\sum_i h_{i,\ w} \sum_0^{u_e} \frac{m_{ij}}{\tau_j} (u_e - u_j) = 0$$

That is, as long as recombination takes place before the dissociated products diffusing toward the wall can reach the wall, the dissociation has no effect on the heat transfer at the wall. This is true even though much of the dissociated products diffuse outward, thus remaining dissociated and providing a net cooling at the outer portion of the boundary layer. For simplicity, the example considered a single plane of dissociation and a single plane of recombination, but the conclusion can be seen to be valid even for distributed regions of dissociation and recombination by approximating the distributed regions with many small sources and sinks rather than just one of each. At each of these planes of reactions, the summation term for H_2, and therefore the corresponding terms for the other reacting species, will be zero, so that the heat transfer to the surface remains unaffected by the dissociation and recombination.

Concluding Remarks

The analysis presented herein has treated the effect of chemical reactions within a laminar boundary layer for which it was postulated that the Prandtl and Lewis moduli are

unity, and that chemical reactions are restricted to a narrow zone and proceed at an infinite rate. Under these conditions it was found that:

1 If the reaction is irreversible and the critical reaction temperature is less than the wall temperature, the reaction is limited by the diffusion of reactant from the external stream, and the effect upon the heat transfer to the surface is to change the potential from that of the nonreacting (but blowing) case by an amount equal to the heat of reaction per unit mass of the external stream, independent of the location of the reaction plane.

2 If the reaction is irreversible and the critical temperature is greater than the wall temperature, and in addition the blowing rate is less than a certain critical rate (Equation [45]), the reaction is temperature-limited and the effect is to change the heat transfer potential in a manner similar to case 1 but by a lesser amount, the decrease depending on the distance of the reaction from the surface (Equation [47]). For greater than critical blowing, the reaction is no longer temperature-limited but will be limited by diffusion of the reactant from the external stream, as in case 1, and the results are identical to that of case 1. The solution is similar for an irreversible dissociation except that in this case there is no critical blowing rate.

3 If the reaction is reversible but recombination takes place such that no reaction products reach the wall, there is no effect of the reaction on the heat transfer potential, independent of the location of the dissociation and recombination within the boundary layer.

The present analysis does not examine the effects of departure from $\overline{Pr} = 1$ and/or $Le_i = 1$. One method for determining these effects would be to expand the pertinent equations (e.g., [5, 6]) in terms of $(1 - \overline{Pr})$ and $(1 - Le_i)$ so that one could calculate perturbation solutions. In view of the possible import of these departures and the inability at present to carry over the conclusion from nonreacting boundary layer theory that these departures are not significant, it is apparent that the more generalized solution merits further effort. It is of equal importance to define the role played by the rates of chemical processes involved; they must, in actuality, be finite. The ultimate goal of an analysis would be to include these effects as well as those of realistic values of Prandtl and Lewis moduli.

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Heat Transfer in Boundary Layers With Chemical Reactions Due to Mass Addition

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A review is presented of the essential features of the differential equations applicable in the boundary layer with mass transfer at the wall, including consideration of chemical reactions. A brief derivation is given, and the equations are presented in the alternate forms useful for various applications. Special discussion is given of the wall boundary conditions because of possible confusion arising from effects of mass transfer at the wall. A general formal solution to these equations is obtained which makes it possible to determine the heat transfer potential, subject only to the requirement that Lewis number equal unity, for all possible boundary layer chemical reactions; no limitation is placed on reaction rates or other details of the reaction, or on the Prandtl number. In the limiting case where reactions go to completion, the potential may be written directly. As examples, solutions are presented for the case of combustion of carbon to CO and for the dissociation and recombination within the boundary layer of an injected material.

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Nomenclature

C_i	=	concentration by mass of the ith species, ρ_i/ρ
c_p	=	specific heat at constant pressure, $(\partial h/\partial T)_p$
$\bar{c}_p$	=	"frozen" specific heat, $\bar{c}_p = \sum_i C_i c_{p_i} = \left(\frac{\partial h}{\partial T} - \sum_i h_i \frac{\partial C_i}{\partial T}\right)_p$
ΔC_i	=	$C_i - C_{i,w}$
D_i	=	mass diffusion coefficient of the ith species in the total gas mixture
E	=	internal energy
f	=	function defined by Equation [25]
h	=	enthalpy of mixture, $\sum_i C_i h_i$
h'	=	$\sum_i C_i h_i' = \sum_i C_i (h_i - h_{i,w})$
Δh_r	=	heat of reaction
$\Delta h_{r,j,1,w}$	=	heat of reaction of chemical reaction j, per unit mass of species 1 of reaction j, evaluated at wall conditions
Δh_S	=	heat of sublimation
ΔH	=	heat transfer potential, defined by Equation [1]
k	=	"frozen" thermal conductivity

L = operator, defined by Equation [19]
Le_i = Lewis number of the ith species, $\rho D_i \bar{c}_p/k$
n = molar coefficient in balanced chemical equation
O, CO, C, etc. = chemical species
p = pressure
Pr = "frozen" Prandtl number, $\mu \bar{c}_p/k$
q_w = heat transfer rate per unit area at the wall,

$$q_w = \left(k\frac{\partial T}{\partial y}\right)_{g,w} = \left(\frac{\mu}{Pr}\frac{\partial h'}{\partial y}\right)_{g,w}$$

Q = heat transferred
r = recovery factor
t = time
T = temperature
u, v = velocity components in the x and y directions
V = volume
w_i = mass source rate per unit volume of the ith species
x, y = longitudinal and normal coordinate
X = chemical species entering reaction
α = reciprocal of formula weight (product of molecular weight and molar coefficient)
μ = dynamic viscosity
ρ = mass density
ϕ = dissipation function
ϕ' = $\phi + u\dfrac{\partial p}{\partial x}$

Subscripts

e = boundary layer edge
g = gas phase
i = refers to chemical species or energy mode of a single species
I = refers to nonreacting species
j = identifies the chemical reaction
O = oxygen, or per unit mass of oxygen
r = reaction
S = surface material, or per unit mass of surface material
w = at the wall, or at the wall condition
1, 2, etc. = single number identifies species
$j, i; j, 1; j, 2$; etc. = first number identifies the chemical reaction, second identifies species in the reaction

Introduction

SOME schemes for protection against high rates of heat transfer for high performance airplanes and missiles involve injecting foreign gas into the boundary layer or ablation of the surface material. The opportunity is great for this added material to react with the boundary layer gases, or to undergo dissociation or other changes involving release or absorption of energy, with the possibility of appreciably affecting the heat transfer to the surface. Several authors have considered various aspects of the chemically reacting boundary layer, for example Hirschfelder (1)[3] and Dooley (2); the emphasis in this paper is on those reactions involving gas injected through or vaporized from the surface. Reference 3 also treats this problem but with a different and more restrictive approach.

If the equation for heat transfer to a surface is written

$$q_w = c_h \rho_e u_e \Delta H \quad \ldots\ldots [1]$$

it is quite useful to examine the Stanton number c_h and heat transfer potential ΔH, separately. It has been demonstrated by many authors that c_h has a relatively weak dependence on the details of the boundary layer profile (providing that some approximate account is taken of the property variations, as for example by evaluation at an appropriate reference enthalpy). Therefore if, as shown later, the potential itself can be determined in a rather general form, and if c_h can be determined to a sufficiently good approximation utilizing existing solutions for the nonreacting situation (including the blowing effect of the gas entering the boundary layer from the surface, now available from solutions by various authors), we have immediately at our disposal useful working relations for the calculation of heat transfer with chemical reactions and gas injection.

In (3) the heat transfer potential with chemical reactions is obtained, subject to the restrictions of Prandtl number equal to unity and zero pressure gradient, by determining the detailed boundary layer profiles in terms of the velocity, using the Crocco transformations. In this paper a more general approach is used to obtain the potential for particular cases. The Lewis number is restricted to unity, but the Prandtl number and pressure gradient are unrestricted.

In addition to examining certain features of the governing equations considerable attention is given, as noted above, to the determination of the heat transfer potential with chemical reactions and surface gas injection.

It is believed that the derivations leading to the final formulation, Equation [36], for the potential are new; however, except for the special formulations given, the specific results of the examples presented are not completely original, see for example, (3).[4]

Brief Derivations of Appropriate Boundary Layer Equations

Energy Equation

The energy equation describing the compressible boundary layer in the absence of chemical reactions and cross diffusion effects is[5]

$$\rho u\frac{\partial h}{\partial x} + \rho v\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \phi + u\frac{\partial p}{\partial x} \quad \ldots\ldots [2]$$

where we have dropped certain terms as in the usual boundary layer approximations, and ϕ is the energy addition per unit volume by viscous dissipation. It will be convenient in the remainder of the paper to combine the last two terms into a single term, ϕ'. For the steady-state boundary layer, the equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad \ldots\ldots [3]$$

Now, keeping in mind that $k(\partial T/\partial y)$ refers to heat transfer by molecular exchange processes and does not include any energy transfer accompanying mass transfer (even of a single species in the case of a multicomponent gas), we can immediately write the energy equation for a gas composed of many species

$$\sum_i \rho_i u_i\frac{\partial h_i}{\partial x} + \sum_i \rho_i v_i\frac{\partial h_i}{\partial y} + \sum_i w_i h_i = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \phi' \quad \ldots\ldots [4]$$

where w_i is the mass rate of chemical formation per unit volume of species i, and where $w_i h_i$ represents the rate at which enthalpy is "soaked up" by such formation.[6] Note that this term acts like the term convecting enthalpy out and is thus added to the terms on the left. Any effects of chemical reactions will be associated with the species conversions

[3] Numbers in parentheses indicate References at end of paper.

[4] Similar results have recently been obtained by L. Lees in "Convective Heat Transfer with Mass Addition and Chemical Reactions," Third Combustion and Propulsion Colloquium AGARD, NATO, Palermo, Sicily, March 1958.

[5] The equations here are written for a flat plate; the extension to a curved cylindrical surface or body of revolution is easily made.

[6] The subscript i is normally taken to indicate the chemical species, but more generally can be taken, without change in what follows, to distinguish between various energy states of a given species as well as between species.

(or mass production) represented by the w_i terms. When account is taken of all reactions only Δh's of formation (+ and −) will appear in the term $\sum_i w_i h_i$. The k used here is the "frozen" value for the gas mixture (4). $\rho_i u_i$ and $\rho_i v_i$ are the existing mass fluxes in the x and y directions for the species i, whatever might be the causes of such flow (convection or diffusion).

The species continuity equation is

$$\frac{\partial}{\partial x}(\rho_i u_i) + \frac{\partial}{\partial y}(\rho_i v_i) = w_i \quad \ldots\ldots [5]$$

and $\sum_i w_i = 0$, so that Equation [3] still applies.

It is convenient at this point to introduce a particular reference value for the enthalpy of each species. We define a quantity

$$h_i' = h_i - h_{i,w}$$

where $h_{i,w}$ is the enthalpy of species i at wall conditions[7] taken to be constant in this analysis, and substitute this into Equation [4] yielding

$$\sum_i \rho_i u_i \frac{\partial h_i'}{\partial x} + \sum_i \rho_i v_i \frac{\partial h_i'}{\partial y} + \sum_i w_i h_i' + \sum_i w_i h_{i,w} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \phi' \quad \ldots\ldots [6]$$

Multiplying [5] by h_i', summing over species i, and adding the resulting Equation to [6] yields

$$\sum_i \frac{\partial}{\partial x}(\rho_i u_i h_i') + \sum_i \frac{\partial}{\partial y}(\rho_i v_i h_i') + \sum_i w_i h_{i,w} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \phi' \quad \ldots\ldots [7]$$

We now write the species mass velocity (flux density) in terms of the mean mass flow plus a diffusion term[8] (relative to the mean flow)

$$\rho_i u_i = C_i(\rho u) - \rho D_i \frac{\partial C_i}{\partial x}$$

and

$$\rho_i v_i = C_i(\rho v) - \rho D_i \frac{\partial C_i}{\partial y} \quad \ldots\ldots [8]$$

where C_i is the concentration (mass fraction) of the ith species. Substituting in Equation [7] we obtain

$$\frac{\partial}{\partial x}(\rho u \sum_i C_i h_i') + \frac{\partial}{\partial y}(\rho v \sum_i C_i h_i') - \frac{\partial}{\partial x}\left(\sum_i \rho D_i \frac{\partial C_i}{\partial x} h_i'\right) - \frac{\partial}{\partial y}\left(\sum_i \rho D_i \frac{\partial C_i}{\partial y} h_i'\right) + \sum_i w_i h_{i,w} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \phi'$$

Transposing terms, noting that $\sum_i C_i h_i' = h'$, and neglecting x derivatives of diffusion terms, we have

$$\frac{\partial}{\partial x}(\rho u h') + \frac{\partial}{\partial y}(\rho v h') + \sum_i w_i h_{i,w} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + \sum_i \rho D_i \frac{\partial C_i}{\partial y} h_i'\right) + \phi' \quad \ldots\ldots [9]$$

or, on subtracting the total (global) continuity Equation [3]

$$\rho u \frac{\partial h'}{\partial x} + \rho v \frac{\partial h'}{\partial y} + \sum_i w_i h_{i,w} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + \sum_i \rho D_i \frac{\partial C_i}{\partial y} h_i'\right) + \phi' \quad \ldots\ldots [9a]$$

It is important to note that by introducing the reference enthalpies in this manner a source term, $\sum_i w_i h_{i,w}$, appears, thus introducing the heats of formation in the gas phase evaluated at reference conditions. Also, it is quite clear that all enthalpies enter as differences, taken separately for each species. Inasmuch as the concept of enthalpy is meaningful only in terms of enthalpy differences, species by species, it is useful to have equations (such as [9 and 9a]) where these enthalpy differences are demonstrated explicitly and where no apparent absolute enthalpies appear. The heat of formation is, of course, itself a difference of enthalpies.

Diffusion Equation

If Equations [8] are substituted into the species continuity Equation [5], the expression

$$\frac{\partial}{\partial x}(C_i \rho u) + \frac{\partial}{\partial y}(C_i \rho v) = \frac{\partial}{\partial y}\left(\rho D_i \frac{\partial C_i}{\partial y}\right) + w_i \quad \ldots\ldots [10]$$

is obtained, dropping the x derivatives of the diffusion term.

This may be combined with the total continuity Equation [3] to yield the alternate form

$$\rho u \frac{\partial C_i}{\partial x} + \rho v \frac{\partial C_i}{\partial y} = \frac{\partial}{\partial y}\left(\rho D_i \frac{\partial C_i}{\partial y}\right) + w_i \quad \ldots\ldots [11]$$

In general, in addition to the diffusion equations, the chemical equations and rate reaction equations are necessary to determine the w_i terms.

Momentum Equation

The momentum equation for the boundary layer is not altered by mass diffusion, except insofar as the density and viscosity are affected by the concentration changes, and remains as

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} \quad \ldots\ldots [12]$$

[7] The choice of reference value is a matter of convenience. Choosing wall values simplifies the statement of boundary conditions because $h_i' = 0$ for all species at the wall.

[8] We have neglected diffusion due to all gradients except species concentration gradients (thermal diffusion, pressure diffusion, etc.). Also, note that the D_i's are the same as the "number" diffusion coefficient used in kinetic theory. See for example (6). The D_i for a particular species is the diffusion coefficient for species i in a background of all other species present. It can be computed by resorting to complex analytical techniques, and in principle can be measured experimentally (by suitable sampling, plus chemical or mass spectrographic techniques) in the actual flow situation.

Boundary Conditions

Energy Equation

The boundary conditions on the energy equation warrant special discussion. The wall boundary condition is readily understood by use of Equations [4 and 5]. If Equation [5] is multiplied by h_i and summed over i and then combined with [4], thus eliminating the "source" term, then the equation applicable for the region near the wall can be written (because all $u_{i,w} = 0$) as

$$\frac{\partial}{\partial y}\left(\sum_i \rho_i v_i h_i\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \phi' \quad \ldots\ldots [13]$$

Keeping in mind that a coordinate system has been chosen (implicitly) in which the surface is fixed and mass flows through it (including the wall material flowing through the surface) Equation [13] can most conveniently be integrated across the interface by employing a fictitious boundary a small distance ϵ out from the wall in the gas stream, such that no reactions occur between the boundary and the wall. Clearly, for any reactions not involving the wall material, for example recombination of atomic species from the external flow, the temperature and property variations are so small over the interval that the contribution to the wall heat transfer will be the same whether the process occurs on the wall or a distance ϵ away from it. (The kinetics of the reaction, on the other hand, may well depend on whether it is a surface reaction or not.) Similarly, reactions involving the wall material behave the same thermodynamically whether the reaction is at the fictitious boundary or on the wall. That is, in one case vaporization occurs first, and the reaction takes place in the gas phase at the distance ϵ from the wall; in the other case the reaction occurs on the wall in the solid phase, and the gaseous product leaves the wall and crosses the fictitious boundary; the end result is the same regardless of the path of the process as long as the end temperatures are the same.[9] In summary, all surface reactions behave as gas phase reactions very near the surface and can be so treated, at least as far as heat transfer considerations are concerned.

Now letting the fictitious boundary approach the wall as $\epsilon \to 0$ and integrating across the interface yields

$$\left(\sum_i \rho_i v_i h_i\right)_{g,w} - (\rho v)_w h_{\text{solid},\,w} = \left(k\frac{\partial T}{\partial y}\right)_{g,w} - \left(k\frac{\partial T}{\partial y}\right)_{\text{solid},\,w}$$

(the contribution of ϕ' is zero when integrated across the interface). Because we have in effect transferred the surface reactions to the gas region slightly away from the surface, the only mass flow at the interface is that of the surface material. The left side of the above equation then becomes $(\rho v)_w$ $(h_{g,w} - h_{\text{solid},\,w})$ where the enthalpy difference is the heat of sublimation, Δh_S. From the following section (Equation [18]) it can be seen that at the wall, where $h'_{i,w} = 0$,

$$\left(k\frac{\partial T}{\partial y}\right)_{g,w} = \left(\frac{\mu}{Pr}\frac{\partial h'}{\partial y}\right)_{g,w}$$

It is this quantity which will be referred to as the wall heat transfer rate, q_w; in the discussions to follow, special emphasis will be placed on its evaluation. We can now write the boundary condition in a form more convenient for use here as[10]

$$q_w = \left(\frac{\mu}{Pr}\frac{\partial h'}{\partial y}\right)_{g,w} = (\rho v)_w \Delta h_S + \left(k\frac{\partial T}{\partial y}\right)_{\text{solid},\,w} \quad \ldots\ldots [14]$$

When writing the edge boundary condition, it should be recalled that the enthalpy for a particular species at the edge of the boundary layer is not constant; rather, the total enthalpy including any chemical energy due to species conversions is constant at the edge.

Diffusion Equation

For the surface material

$$(\rho_i v_i)_{i=S,w} = (\rho v)_w$$

From Equation [8], we have at the wall

$$(\rho_i v_i)_w = (C_i \rho v)_w - \left(\rho D_i \frac{\partial C_i}{\partial y}\right)_w$$

so that

$$(\rho v)_w (1 - C_{S,w}) = -\left(\rho D_S \frac{\partial C_S}{\partial y}\right)_w \quad \ldots\ldots [15]$$

For all other species, excluding those reacting at the wall, $(\rho_i v_i)_w = 0$ and

$$(C_i \rho v)_w = \left(\rho D_i \frac{\partial C_i}{\partial y}\right)_w \quad \ldots\ldots [16]$$

Development of Alternate Form

For some purposes it is desirable to write Equations [9 and 11] in different forms. In particular, the introduction of ratios of diffusivities (the Lewis and Prandtl numbers) is useful. We note that[11]

$$\frac{\partial h'}{\partial y} = \frac{\partial}{\partial y}\left(\sum_i C_i h_i'\right) = \sum_i C_i \frac{\partial h_i'}{\partial T}\frac{\partial T}{\partial y} + \sum_i h_i' \frac{\partial C_i}{\partial y}$$

$$= \bar{c}_p \frac{\partial T}{\partial y} + \sum_i h_i' \frac{\partial C_i}{\partial y} \quad \ldots\ldots [17]$$

where $\bar{c}_p$ is the "frozen" heat capacity (that is, the heat capacity summed over species i and not accounting for species production or conversion due to chemical reactions). Then the first term on the right in Equation [9] can be written

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{k}{\mu \bar{c}_p}\mu \bar{c}_p \frac{\partial T}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{k}{\mu \bar{c}_p}\mu \frac{\partial h'}{\partial y} - \frac{k}{\mu \bar{c}_p}\mu \sum_i \frac{\partial C_i}{\partial y} h_i'\right)$$

so that the right side of [9] becomes

$$\frac{\partial}{\partial y}\left\{\frac{\mu}{Pr}\frac{\partial h'}{\partial y} + \frac{\mu}{Pr}\sum_i\left[(Le_i - 1)\frac{\partial C_i}{\partial y} h_i'\right]\right\} + \phi' \quad \ldots [18]$$

where we have defined "frozen" Prandtl number in terms of "frozen" k and $\bar{c}_p$, and $Le_i = \rho D_i \bar{c}_p / k$.

Arguments can be made for using the regular Prandtl number, using the complete c_p and k (including mass diffusion-carried energy) as is done, for example, in (1). However, use of the frozen properties has the virtue of allowing easy identification of the phenomena involved in the diffusing and reacting boundary layer.

It will be helpful to adopt the notation

$$L(\ ) = \rho u \frac{\partial(\ )}{\partial x} + \rho v \frac{\partial(\ )}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial(\ )}{\partial y}\right) \quad \ldots [19]$$

The diffusion Equation [11] now becomes (after replacing

[9] A more direct, but somewhat more involved, approach for handling surface reactions is to treat them as they actually occur, on the surface, in which case the result of integrating Equation [13] across the interface is to yield terms of the form $(\rho v)_i \Delta h_{r,i,w}$ by the methods of the section on "Determination of the Heat Transfer Potential," for both recombinations and reactions involving the surface material. The factor $(\rho v)_i$ is then evaluated using Equation [8] and the conditions imposed in the above named section to yield terms for surface reactions identical to those for gas phase reactions (see Equation [36]).

[10] Radiation effects have been neglected throughout.

[11] Procedurally, we are following the scheme used by Lees (4) in using $\bar{c}_p$ and the "frozen" Prandtl number.

C_i by $C_i - C_{i,w} = \Delta C_i$, where $C_{i,w}$ is a constant)

$$L(\Delta C_i) \equiv \rho u \frac{\partial \Delta C_i}{\partial x} + \rho v \frac{\partial \Delta C_i}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\mu}{\overline{Pr}} \frac{\partial \Delta C_i}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{\mu}{\overline{Pr}} (Le_i - 1) \frac{\partial \Delta C_i}{\partial y}\right] + w_i \quad [20]$$

and the energy Equation [9a] becomes

$$L(h') \equiv \rho u \frac{\partial h'}{\partial x} + \rho v \frac{\partial h'}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\mu}{\overline{Pr}} \frac{\partial h'}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{\mu}{\overline{Pr}} \sum_i h_i' (Le_i - 1) \frac{\partial \Delta C_i}{\partial y}\right] - \sum_i w_i h_{i,w} + \phi' \quad [21]$$

The source term may now be eliminated between Equations [20 and 21] by multiplying [20] through by $h_{i,w}$, summing over all i's and adding to Equation [21]. There results[12]

$$L\left(h' + \sum_i \Delta C_i h_{i,w}\right) = \frac{\partial}{\partial y}\left[\frac{\mu}{\overline{Pr}} \sum_i (Le_i - 1) \frac{\partial \Delta C_i}{\partial y} (h_i' + h_{i,w})\right] + \phi' \quad [22]$$

As before, enthalpies appear as differences only (it will be shown later that the term involving $\sum_i h_{i,w}$ in Equation [22] represents the heats of the chemical reactions which occur).

Determination of the Heat Transfer Potential (Case of $Le_i = 1$)

Without completely solving the various equations it is possible to obtain the form of the heat transfer potential, and this will now be done.[13] Because of the difficulty involved in handling, in a simple way at least, the case of $Le_i \neq 1$, only the case where this parameter is unity will be presented. Although one can argue, from previous experience with the nondiffusing and nonreacting boundary layer, that the effects of $\overline{Pr} \neq 1$ are not appreciable and can be approximately accounted for by simple formulations, it is not clear that this will be true for $Le_i \neq 1$. The question of the importance of such deviations must be pursued, possibly by perturbation methods—or possibly by machine calculations.

In the examples to be given ϕ' is taken as zero with the idea that flow work and dissipation effects can be added to the heat transfer potential by use of an appropriate recovery factor times $u_e^2/2$. On taking $Le_i = 1$ the combined energy Equation [22] becomes

$$L\left(h' + \sum_i \Delta C_i h_{i,w}\right) = 0 \quad [23]$$

The argument of this equation may be normalized so that the limit values are zero at the wall and unity at the edge of the boundary layer. Then we have

$$L\left(\frac{h' + \sum_i \Delta C_i h_{i,w}}{h_e' + \sum_i \Delta C_{i,e} h_{i,w}}\right) = 0 \quad [24]$$

for the energy equation. The solution to this is some function $f(x, y)$ where

$$L(f) = 0 \quad [25]$$

and f is zero at the wall and unity at the edge. We now have

$$h' + \sum_i \Delta C_i h_{i,w} = f\left(h_e' + \sum_i \Delta C_{i,e} h_{i,w}\right) \quad [26]$$

Differentiating with respect to y and evaluating at the wall yields

$$\left(\frac{\partial h'}{\partial y}\right)_{e,w} = \left(\frac{\partial f}{\partial y}\right)_{e,w} \left\{h_e' + \sum_i h_{i,w}\left[\Delta C_{i,e} - \left(\frac{\partial C_i}{\partial f}\right)_w\right]\right\} \quad [27]$$

It is fruitful at this point to introduce the chemical reaction equation, or set of equations, representing the reaction or reactions occurring in the boundary layer. These equations are of the form

$$\sum_j \sum_i n_{j,i} X_{j,i} = 0$$

where $X_{j,i}$ refers to particular chemical species (as O, CO, O_2, N_2, etc.) and $n_{j,i}$ are the molar coefficients in the chemical equations. (The first subscript j identifies the chemical reaction; the second i identifies the species of the jth reaction.) From these equations can be computed values of the coefficients for the volume mass production rates, such that

$$\alpha_{j,1} w_{j,1} = \alpha_{j,2} w_{j,2} = \alpha_{j,3} w_{j,3} = \ldots$$

where α is the reciprocal of the formula weight (that is, the reciprocal of the product of molecular weight and the molar coefficient). For each chemical reaction, the volume mass production rates of each species of the reaction can be written in terms of the volume mass production rate of one particular species of each reaction

$$w_{j,2} = \frac{\alpha_{j,1}}{\alpha_{j,2}} w_{j,1} \qquad w_{j,3} = \frac{\alpha_{j,1}}{\alpha_{j,3}} w_{j,1}, \text{ etc.}$$

We are now ready to use these relations. Equation [20] can be written, for $Le_i = 1$, as

$$L(\Delta C_{j,1}) = w_{j,1}$$

and

$$L(\Delta C_{i,2}) = w_{j,2} = \frac{\alpha_{j,1}}{\alpha_{j,2}} w_{j,1}, \text{ etc.}$$

so that

$$L\left(\frac{\alpha_{j,2}}{\alpha_{j,1}} \Delta C_{j,2}\right) = w_{j,1}, \text{ etc.}$$

The first and last forms can be subtracted, yielding

$$L\left(\Delta C_{j,1} - \frac{\alpha_{j,2}}{\alpha_{j,1}} \Delta C_{j,2}\right) = 0$$

for which the solution in normalized form is a gain f, as in Equations [24 and 25], and

$$\Delta C_{j,1} - \frac{\alpha_{j,2}}{\alpha_{j,1}} \Delta C_{j,2} = f\left(\Delta C_{j,1,e} - \frac{\alpha_{j,2}}{\alpha_{j,1}} \Delta C_{j,2,e}\right)$$

Differentiating with respect to f and rearranging terms we have at the wall

$$\Delta C_{j,1,e} - \left(\frac{\partial C_{j,1}}{\partial f}\right)_w = \frac{\alpha_{j,2}}{\alpha_{j,1}}\left[\Delta C_{j,2,e} - \left(\frac{\partial C_{j,2}}{\partial f}\right)_w\right] \quad [28]$$

etc., so that the quantity required in Equation [27] for any species of reaction j can be expressed in terms of one particular species ($i = 1$) of that reaction.

[12] The term ϕ' is carried in this form and not combined at this point with h' to form $h' + u^2/2$, because it is simpler in the present analysis to add the appropriate terms to the solution later.

[13] Some of the concepts used in the analysis presented here are similar to those used by Zeldovich (5) in an analysis of the flame combustion problem, although details of the analysis here are different. The applicability of these concepts to the boundary layer problem was pointed out to the authors by Lester Lees.

The quantity $(\partial C_i/\partial f)_w$ will later (Equation [32]) be shown to be proportional to wall concentration for any nonsurface species; then the terms for any species which does not appear at the wall or the edge will drop out of the summation of Equation [27]. If then the species entering each reaction are grouped together and the reference species ($i = 1$) of each reaction chosen as any of the species remaining, Equation [27] becomes

$$\left(\frac{\partial h'}{\partial y}\right)_{e,w} = \left(\frac{\partial f}{\partial y}\right)_{e,w}\left\{h_e' + \sum_j\left[\Delta C_{j,1,e} - \left(\frac{\partial C_{j,1}}{\partial f}\right)_w\right]\left(h_{j,1,w} + \frac{\alpha_{j,1}}{\alpha_{j,2}}h_{j,2,w} + \frac{\alpha_{j,1}}{\alpha_{j,3}}h_{j,3,w} + \ldots\right) + \sum_{\substack{i=\text{inert}\\ \text{species}}} h_{i,w}\left[\Delta C_{i,e} - \left(\frac{\partial C_i}{\partial f}\right)_w\right]\right\} \quad \ldots\ldots [29]$$

The enthalpy groupings are simply the heats of each reaction so Equation [29] becomes, on multiplying by $\mu/\overline{Pr}$

$$\left(\frac{\mu}{\overline{Pr}}\frac{\partial h'}{\partial y}\right)_{e,w} = \left(\frac{\mu}{\overline{Pr}}\frac{\partial f}{\partial y}\right)_{e,w}\left\{h_e' + \sum_j \Delta h_{r,j,1,w}\left[\Delta C_{j,1,e} - \left(\frac{\partial C_{j,1}}{\partial f}\right)_w\right] + \sum_{\substack{i=\text{inert}\\ \text{species}}} h_{i,w}\left[\Delta C_{i,e} - \left(\frac{\partial C_i}{\partial f}\right)_w\right]\right\} \quad \ldots\ldots [30]$$

From the usual nonreacting boundary layer theory, by definition of the Stanton number c_h

$$\left(\frac{\mu}{\overline{Pr}}\frac{\partial f}{\partial y}\right)_{e,w} = c_h\rho_e u_e \quad \ldots\ldots [31]$$

It is clear that to determine c_h in the presence of chemical reactions, due account must be taken of the effects of variations in ρ and the transport properties to determine the entire flow field.

It remains to evaluate the terms involving $(\partial C/\partial f)_w$. The expanded quantity to be evaluated is

$$\left(\frac{\mu}{\overline{Pr}}\frac{\partial f}{\partial y}\right)_{e,w}\left(\frac{\partial C_i}{\partial f}\right)_w = \left(\frac{\mu}{\overline{Pr}}\frac{\partial C_i}{\partial y}\right)_w$$

For all species except the surface material this becomes, upon employing Equation [16],

$$= \left(\frac{1}{Le_i}C_i\rho v\right)_w = c_h\rho_e u_e B' C_{i,w} \quad \ldots\ldots [32]$$

where B' is the modified blowing parameter, defined by $B' = (\rho v)_w/c_h\rho_e u_e$, and Le_i has been taken equal to unity. The corresponding term for the surface material is

$$c_h\rho_e u_e B'(1 - C_{S,w})$$

If such terms are reinserted in [30] this equation becomes

$$\left(\frac{\mu}{\overline{Pr}}\frac{\partial h'}{\partial y}\right)_{e,w} = c_h\rho_e u_e\left\{h_e' + \sum_j \Delta h_{r,j,1,w}[C_{j,1,e} - (B'+1)C_{j,1,w}] + \sum_j \Delta h_{r,j,S,w}[B' - (B'+1)C_{j,S,w}] + \sum_{\substack{i=\text{inert}\\ \text{species}}} h_{i,w}[C_{i,e} - (B'+1)C_{i,w}] + h_{S',w}[B' - (B'+1)C_{S',w}]\right\} \quad \ldots\ldots [33]$$

where S' represents nonreacting surface material.

The significance of the terms in square brackets in Equation [33] can be seen by noting that for an inert gas the wall concentration is

$$C_{i,w} = C_{i,e}/(B'+1) \quad \ldots\ldots [34]$$

or, for the surface material when it is inert

$$C_{S,w} = B'/(B'+1) \quad \ldots\ldots [35]$$

(To show these relations, Equations [20 and 25], written for an inert material, lead to

$$\Delta C_I/\Delta C_{I,e} = f$$

where I denotes an inert species. From the definition of c_h

$$\left(\frac{\mu}{\overline{Pr}}\frac{\partial f}{\partial y}\right)_w = c_h\rho_e u_e$$

thus

$$\left(\frac{\mu}{\overline{Pr}}\frac{1}{\Delta C_{I,e}}\frac{\partial \Delta C_I}{\partial y}\right)_w = c_h\rho_e u_e$$

On introducing Equation [16 or 15], setting $Le_I = 1$, and manipulating algebraically, Equation [34 or 35] results.) The inert species therefore drop out of [33], and it becomes

$$q_w = \left(\frac{\mu}{\overline{Pr}}\frac{\partial h'}{\partial y}\right)_{e,w} = c_h\rho_e u_e\left\{h_e' + r\frac{u_e^2}{2} + \sum_j \Delta h_{r,j,1,w}[C_{j,1,e} - (B'+1)C_{j,1,w}] + \sum_j \Delta h_{r,j,S,w}[B' - (B'+1)C_{j,S,w}]\right\} \quad \ldots\ldots [36]$$

where the second summation applies only to those reactions, if any, where the surface material is chosen as the reference. Note that the quantity ϕ' has now been taken into account in Equation [36] by including in the heat transfer potential the term $r(u_e^2/2)$, where r is the recovery factor and is equal to unity when $\overline{Pr} = 1$. In view of relation [34] the quantity $(B'+1)C_{j,1,w}$ in Equation [36] is the edge of boundary layer concentration of the fraction of species 1 which reached the wall without reacting. The bracketed term is then the expected result: Simply the fraction of species 1 (measured at the boundary layer edge) which reacts, with heat release $\Delta h_{r,j,1,w}$.

Another view of these final equations is gained by taking all terms with the factor B' to the other side of the equation; since $B'c_h\rho_e u_e = (\rho v)_w$ these terms combine (recall Equation [14]) with the latent heat of sublimation term. The effect, in other words, of an outer edge species not reacting completely but having a nonzero concentration at the wall, is as though the reaction did go to completion but that the heat of sublimation of surface material was modified accordingly to account for the incompleted part of the reaction.

We have now determined that the heat transfer potential is the usual value, *involving only edge species*, plus the term or terms representing the heats of formation for the reacting species, whether the reactions occur in the gas phase or on the wall, and whether they involve the wall material or are simply recombinations or dissociations of external or product

species. In determining the edge species enthalpies, it is important to note, as mentioned in the section on Boundary Conditions, that the total energy including appropriate chemical energy terms is conserved across any shock waves; furthermore, it is constant along the boundary layer edge.

It is important to remember that the expression for the potential (bracketed term in [36]) is purely formal in that, generally, the concentrations at the wall are not known a priori. To determine the necessary wall concentrations requires (as has been noted earlier) solving the coupled equations describing the energy flow, momentum, species continuity and reaction rates. This means that due account must be taken of the detailed kinetics of each situation; for example, in the determination of the wall concentration (as contrasted with the heat transfer potential) it makes a difference whether reactions actually take place on the surface or a short distance off the surface. It is not the purpose of this paper to discuss the questions of finding the concentrations. Also it should be noted that for the limiting situation of sufficiently fast reaction rates (by which is meant here that the reaction eventually goes to completion somewhere in the boundary layer or on the wall) certain solutions can be obtained as given in the next paragraph.

The application of the above equations to two simple cases can be illustrated:

1 The irreversible combustion on the surface or within the boundary layer of subliming carbon, as given by

$$C + O \rightarrow CO$$

under conditions that all the oxygen is consumed: Taking Equation [36] to be written for oxygen, then $C_{j,1,w} = C_{O,w} = 0$ and $C_{j,1,e} = C_{O,e}$. The solution for the chemical reaction portion of the heat transfer potential is then $C_{O,e}\Delta h_{r,O,w}$ which is the gas phase heat of reaction per unit mass of *air* evaluated at wall conditions. The effect of the reaction is simply to increase the usual enthalpy potential by this amount. The gas phase heat of reaction is the appropriate one here even for surface reactions, because the latent heat of sublimation is included on the other side of the equation (see Equation [14]). If there are intermediate reactions, such as the formation of O_2 from atomic oxygen and the subsequent reaction of O_2 with carbon to form CO, the result is clearly the same as the direct reaction of oxygen atoms with carbon. In such cases it is simpler to treat the reactions in the latter form, referencing the reaction to the outer edge species, atomic oxygen, since its concentration is known at both limits, wall and edge.

2 The dissociation in the boundary layer of a surface or injected material under conditions such that no dissociated products reach the wall (recombination takes place prior to reaching the wall): Recall that Equation [36] does not allow using as a reference species one which disappears at both limits, wall and edge, as do the dissociated products. Referencing both reactions (dissociation and recombination) to the surface material, the appropriate chemical reaction terms of Equation [36] are clearly equal and of opposite sign. It then follows that the enthalpy potential is completely unaffected by the dissociation and recombination, regardless of the location of these reactions in the boundary layer. If recombination is incomplete because of slow recombination rates, a net effect will be felt at the wall depending on the quantity of dissociated products which reaches the wall, which in turn depends on the reaction rates or detailed spatial description of the reactions. If recombination is incomplete because the dissociated products do not diffuse to the wall, the effect on the heat transfer potential is also not zero, but the requirement of $Le_i = 1$ precludes this occurrence here.

It is interesting to note that for the two examples given the results are actually independent of any possible pressure gradient effects. Also, they should apply equally well to turbulent flow, presuming that the same arguments can be made regarding diffusion phenomena as have been shown empirically to apply to heat transfer effects in the nonreacting boundary layer.

The approach presented here thus enables the heat transfer potential in the presence of chemical reactions to be written directly, without the necessity of solving the differential equations, for the class of problems where one of the reactants is consumed completely in the reaction. For all other cases one must either know the blowing rate B' and the wall concentration of the reference species, or the fraction of the edge concentration of the reference species which has reacted irreversibly before diffusing to the wall. In general then, one requires knowledge of the reaction rates.

Conclusions

By examining the differential equations of the boundary layer which allow for diffusion of reacting species and by retaining enthalpy differences species by species and utilizing the concept of "frozen" specific heat, it is possible to identify the changes in the heat transfer potential due to chemical reactions of any nature, involving outer flow gas and/or surface material. Further, by incorporating species conservation relations of the reacting species into the energy equation, certain details of the heat transfer processes become apparent without completely solving the differential equations.

For those cases where one of the reactants is consumed completely by the reaction, the added effects of the chemical reaction on the heat transfer potential may be obtained directly. Two examples illustrate the method: (1) For the irreversible combustion of carbon to CO, it was shown that the heat transfer potential is simply the usual one plus the heat evolved by the reaction, independent of the location of the reaction, and (2) for dissociation and recombination within the boundary layer of an injected material, it is shown that the heat transfer potential is completely unaffected. For all other cases, either the blowing rate and reference species wall concentration must be known, or the fraction of the edge concentration of a reacting species which reacts irreversibly must be known.

The general analysis leading to Equation [22] applies to the flat plate in laminar flow without restriction on Lewis number; the two examples cited follow from Equation [36] and strictly apply only to the case of $Le_i = 1$, but no restriction has been placed on $\overline{Pr}$ or pressure gradients. Although it has not been discussed fully, the general form of the solution should be applicable to turbulent as well as laminar flow by extension of the present work in the manner in which previous laminar analyses have been applied to turbulent flows.

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On the Brachistochronic Thrust Program for a Rocket Powered Missile Traveling in an Isothermal Medium[1]

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The burning program for a rocket powered missile moving along a rectilinear path is investigated with the idea of minimizing the time interval necessary to reach a given point in space. The induced drag is assumed negligible with respect to the zero-lift drag; the atmosphere is regarded as isothermal. By using the indirect methods of the Calculus of Variations, it is shown that the totality of extremal arcs is composed of zero-thrust sub-arcs, sub-arcs flown with maximum engine output, and variable thrust sub-arcs. Concerning the latter, and in connection with suitable hypotheses for the drag function, closed form solutions are obtained for the relationships between altitude, mass, time and Mach number. The boundary value problem is investigated for prescribed initial and final values of altitude, Mach number and mass. The analysis shows that, for given end conditions, two extremizing trajectories exist: One which *minimizes* the time and another one which *maximizes* the time. Attention is also given to the case where the final Mach number is free of choice. A numerical example shows that, assuming the initial Mach number is zero, the optimum path includes: An initial sub-arc flown with maximum engine output, an intermediate sub-arc flown with variable thrust, and a final sub-arc flown by coasting.

Introduction

WHEN a missile is flying in a vertical plane, two possibilities exist for modifying its trajectory: One is to act on the controls of the elevator; the other is to act on the controls of the engine. For a given set of initial conditions, there is an infinite number of trajectories which are mathematically and physically possible, depending upon the way the elevator or the engine or both are being operated as a function of the time. It is in this sense that a missile, constrained to travel in a vertical plane, may be referred to as a machine having two degrees of freedom: One is associated with the elevator's position; the other one is associated with the power setting. Of course, it is assumed that the thrust direction, relative to the airframe or to the vector velocity, is given.

The problem of the optimum operational performance in a vertical plane consists in saturating these two degrees of freedom so that the difference $\Delta G = G_f - G_i$ between the final and initial values of an arbitrarily specified function G of the generalized coordinates of the missile is a minimum.

In view of the analytical difficulties associated with the previously indicated problem, the attention of most researchers has been attracted by comparatively simpler questions. These questions have been essentially originated by freezing one degree of freedom and studying the optimization of the other. For the category of problems of optimum burning program, the degree of freedom associated with the elevator's control is frozen by imposing some geometric or aerodynamic constraint on the trajectory (i.e., vertical flight or level flight or zero-lift trajectory, etc.); the other degree of freedom (the one associated with the maneuver of the engine control) must be determined so as to minimize the quantity ΔG. Previous analyses in this area of questions are now mentioned. Horizontal paths have been investigated in (1 to 3).[3] Vertical trajectories have been considered in (4 to 10). Moreover, near vertical paths have been analyzed in (11) in connection with thermal problems for the missile skin.

Object of the Present Investigation

The present research falls under the broad domain of the problems of optimum burning program. Trajectories which are rectilinear and either vertical or near vertical are considered. An ideally isothermal atmosphere is assumed, in so far as this seems to be the only case, having engineering importance, where solutions in a closed form are possible, thus leading to clear cut information on the qualitative nature of the optimum thrust programming. The brachistochronic problem ($G = t$) is investigated: It is the problem of the minimum time required to reach a given point in space.

Use is made of a special coordinate system, described by Equations [5 to 9]. Such a system embodies a number of distinctive features: It enables one to carry out an almost straightforward determination of the optimizing condition; it is such that all Lagrange multipliers are constant along the variable thrust sub-arc of the extremal solution; it is specially advantageous for the solution of the boundary value problem associated with the brachistochronic path; by the use of Green's theorem (13) it enables one to carry out a sufficiency proof for the solution, in a case where the classical variational criteria due to Legendre-Clebsch and Weierstrass fail to produce any useful information on the minimal or maximal nature of the extremal path.

The latter feature of this new coordinate system deserves to be underlined. As the analysis of (13) shows, for a given set of end conditions there are always two trajectories which yield a stationary value for the time: One is a solution of minimum time; the other one is a solution of maximum time. It is, consequently, of engineering interest to have an analytical method available by which *minimum time* paths can be discerned from *maximum time* trajectories.

Fundamental Hypotheses and Equations of Motion

In this paper use is made of the following assumptions:

1 The rocket powered vehicle is regarded as a particle of mass variable with the time.

2 The thrust is tangent to the flight path.

3 The equivalent exit velocity of the rocket engine is regarded as a constant, independent of the operational condition of the engine.

4 The engine is capable of delivering all mass flows (β) bounded between a lower value ($\beta = 0$) and an upper

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[1] The present paper is a condensed form of the investigation described in reference (13).

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[3] Numbers in parentheses indicate References at end of paper.

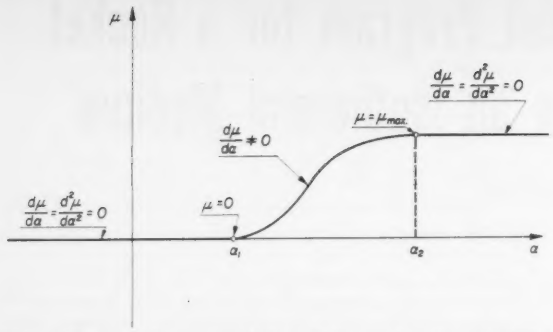

Fig. 1 Parametric representation of the engine characteristics

value ($\beta = \beta_{max}$).

5 The aerodynamic lag is disregarded, i.e., the aerodynamic forces are calculated as in unaccelerated flight.

6 The acceleration of gravity is a constant.

7 The atmosphere is isothermal.

8 The trajectory is straight and inclined at an angle θ with repect to a horizontal plane.

9 The induced drag is negligible[4] with respect to the zero-lift drag.

In the light of assumptions 1, 2, 3, 8 and 9 the dynamic behavior of the rocket powered missile is represented with the following set of differential equations

$$\dot{V} + g \sin\theta + \frac{D - \beta V_e}{m} = 0 \quad [1]$$

$$\dot{h} - V \sin\theta = 0 \quad [2]$$

$$\dot{m} + \beta = 0 \quad [3]$$

where

V = velocity
h = altitude above sea level
m = mass
g = acceleration of gravity
θ = path inclination
D = drag
β = engine mass flow
V_e = equivalent exit velocity of the rocket engine

The dot sign denotes derivative with respect to time.

In view of hypotheses 5, 7 and 9 the drag D is represented as

$$D = \frac{\gamma}{2} p_o S C_{DO}(M) M^2 \exp\left(-\gamma \frac{gh}{a^2}\right) \quad [4]$$

where

γ = ratio of specific heat at constant pressure to specific heat at constant volume (air)
p_o = sea level pressure
S = reference surface
a = speed of sound
C_{DO} = drag coefficient at zero-lift, assumed to depend on the Mach number (M) only.

Transformation of the Equations of Motion

The following set of non-dimensional variables is introduced

$$\tau = t \frac{g}{a} \sin\theta \quad [5]$$

$$\eta = h \frac{g}{a^2} \quad [6]$$

$$M = \frac{V}{a} \quad [7]$$

$$\epsilon = \frac{2g \sin\theta}{\gamma p_o S} m \exp\left(\gamma \frac{g}{a^2} h\right) \quad [8]$$

$$\mu = \frac{2\beta a}{\gamma p_o S} \quad [9]$$

These variables are employed to transform Equations [1 to 3] from the (t, h, V, m, β) space into the (τ, η, M, ϵ, μ) space. After assuming the Mach number (M) as the new independent coordinate, extensive manipulations yield these results

$$\tau' + \varphi_1 + \psi_1 \epsilon' = 0 \quad [10]$$

$$J_2 \equiv \eta' + \varphi_2 + \psi_2 \epsilon' = 0 \quad [11]$$

$$J_3 \equiv \mu + \varphi_3 + \frac{\psi_3}{\eta'} = 0 \quad [12]$$

where primed symbols denote derivatives with respect to Mach number

$$\tau' = d\tau/dM \qquad \epsilon' = d\epsilon/dM \qquad \eta' = d\eta/dM$$

The φ_k-functions and the ψ_k-functions ($k = 1, 2, 3$) are defined as

$$\varphi_1 = \frac{1}{1 + \dfrac{C_{DO} M^2}{\epsilon} - \gamma M M_e} = \varphi_1(M, \epsilon) \quad [13]$$

$$\psi_1 = \frac{M_e}{\epsilon} \varphi_1 = \psi_1(M, \epsilon) \quad [14]$$

$$\varphi_2 = M \varphi_1 = \varphi_2(M, \epsilon) \quad [15]$$

$$\psi_2 = M \frac{M_e}{\epsilon} \varphi_1 = \psi_2(M, \epsilon) \quad [16]$$

$$\varphi_3 = -\frac{C_{DO} M^2 + \epsilon}{M_e} \exp(-\gamma\eta) = \varphi_3(M, \epsilon, \eta) \quad [17]$$

$$\psi_3 = -\epsilon \frac{M}{M_e} \exp(-\gamma\eta) = \psi_3(M, \epsilon, \eta) \quad [18]$$

where $M_e = V_e/a$ is the ratio of equivalent exit velocity of the rocket engine to atmospheric speed of sound (M_e, therefore, does *not* stand for Mach number in the exit section of the rocket engine).

It must be recalled that, in view of hypothesis 4, the actual mass flow β is a quantity bounded between a lower value ($\beta = 0$) and an upper value ($\beta = \beta_{max}$). The non-dimensional mass flow μ, therefore, is also bounded between a lower value ($\mu = 0$) and an upper value ($\mu = \mu_{max}$). This information can be translated into the setting of the variational problem by introducing the idea of parametric representation of the engine characteristics, already used by the writer in (3, 9 and 11). The non-dimensional mass flow μ is represented as a function of a parameter α having the following properties (Fig. 1)

$$\text{for } -\infty \leq \alpha \leq \alpha_1: \quad \mu = 0$$

$$\text{for } \alpha_2 \leq \alpha \leq +\infty: \quad \mu = \mu_{max}$$

$$\text{for } \alpha_1 < \alpha < \alpha_2: \quad \frac{d\mu}{d\alpha} \neq 0$$

With this scheme, α is considered the *independent parameter* of the rocket engine and is allowed to vary between $-\infty$ and $+\infty$. The mass flow μ becomes a dependent quantity, varying between 0 and μ_{max}, according to the scheme of Fig. 1. Notice that the condition $d\mu/d\alpha = 0$ represents either a coasting flight or a flight with maximum engine output. On the other hand, $d\mu/d\alpha \neq 0$ represents any other operating condition of the engine between the two limiting ones.

[4] Notice that hypotheses 8 and 9 limit the applicability of the present theory to vertical or near vertical trajectories.

Notice also that α is *only* a parameter and that there is no necessity of attributing to it any special physical meaning.[5]

$$\frac{d}{dM}\left[-\psi_1 + \lambda_2\psi_2\right] = -\frac{\partial\varphi_1}{\partial\epsilon} - \epsilon'\frac{\partial\psi_1}{\partial\epsilon} + \lambda_2\left[\frac{\partial\varphi_2}{\partial\epsilon} + \epsilon'\frac{\partial\psi_2}{\partial\epsilon}\right] + \lambda_3\left[\frac{\partial\varphi_3}{\partial\epsilon} + \frac{1}{\eta'}\frac{\partial\psi_3}{\partial\epsilon}\right] \quad [24]$$

$$\frac{d}{dM}\left[\lambda_2 - \lambda_3\frac{\psi_3}{\eta'^2}\right] = \lambda_3\left[\frac{\partial\varphi_3}{\partial\eta} + \frac{1}{\eta'}\frac{\partial\psi_3}{\partial\eta}\right] \quad [23]$$

Variational Formulation

The time interval $\Delta\tau = \tau_f - \tau_i$ necessary to transfer the rocket powered missile from the initial point to the final point is calculated by integration of Equation [10]. Assuming that the initial time instant is $\tau_i = 0$, one obtains

$$\tau_f = \int_{M_i}^{M_f} J_1 dM \quad [19]$$

where

$$J_1 \equiv -\varphi_1 - \psi_1\epsilon' \quad [20]$$

Attention is now focused on the differential Equations [11 and 12], where φ_2, ψ_2, φ_3, ψ_3 are defined by Equations [15 to 18] and $\mu = \mu(\alpha)$. This set involves one independent variable (M) and three dependent variables (η, ϵ, α). One degree of freedom is left, and an optimum requirement can therefore be imposed.

Concerning the end conditions, it is assumed that the coordinates M_i, η_i, ϵ_i, η_f, ϵ_f are prescribed, the subscript (i) referring to initial point and the subscript (f) denoting final point.[6] With regard to the final Mach number M_f, two cases are considered: The case where M_f is given and the case where M_f is free of choice. Since α' is nowhere present in the expressions for J_1, J_2, J_3, no condition can be imposed on α. The end values for α (i.e., for the mass flow μ) must be determined from the solution of the variational problem. The latter can be treated indifferently within the frame of questions of Lagrange type (12) or questions of Mayer type (12).

If the Lagrange formulation is adopted, the brachistochronic problem is stated as follows: "Among all sets of functions $\eta(M)$, $\epsilon(M)$ and $\alpha(M)$ which are consistent with Equations [11 and 12] and with certain prescribed end conditions, to determine the special set which minimizes the integral [19]".

Euler Equations

A set of Lagrange multipliers $\lambda_1 = 1$, $\lambda_2 = \lambda_2(M)$, $\lambda_3 = \lambda_3(M)$ is introduced and this expression formed

$$F = \sum_{k=1}^{3} J_k\lambda_k \quad [21]$$

where J_1 is the integrand function appearing in Equation [19]; J_2 and J_3 are the first members of the equations representing the constraints of the variational problem.

Since there are three unknown functions, three Euler equations must be written

$$\frac{d}{dM}\left(\frac{\partial F}{\partial z_J'}\right) = \frac{\partial F}{\partial z_J} \qquad (J = 1, 2, 3) \quad [22]$$

In the above equations $z_1 = \eta$, $z_2 = \epsilon$, $z_3 = \alpha$, and F is the so-called fundamental function defined by Equation [21]. Simple manipulations yield the explicit form for the Euler equations

$$0 = \lambda_3\frac{d\mu}{d\alpha} \quad [25]$$

Discontinuity of the Eulerian Solution

As the Euler Equation [25] shows, the extremal arc is discontinuous, being generally composed of sub-arcs of equation

$$\lambda_3 = 0 \quad [26]$$

and sub-arcs of equation

$$\frac{d\mu}{d\alpha} = 0 \quad [27]$$

Equation [26] represents a flight condition with a continuously variable thrust, as shown in the next sections. Equation [27] offers alternatives: Either a coasting flight ($\mu = 0$) or a flight with maximum engine output ($\mu = \mu_{max}$).

Corner Conditions

Owing to the discontinuous character of the solution, the important conditions of Erdmann and Weierstrass must be satisfied (12) at each corner point, i.e., at each junction of different sub-arcs. The details of the analysis are omitted for brevity and can be found in (13). The conclusions are: At each junction point the multipliers λ_2 and λ_3 must be continuous, and the latter must also be zero. This circumstance evidently implies that a corner point *may* be located on the variable thrust sub-arc of the extremal solution.

Boundary Conditions

For the present problem, the boundary conditions include a number of fixed end point conditions plus one natural condition. This must be deduced from the general transversality condition of the Calculus of Variations (12)

$$\left[\left(F - \sum_{J=1}^{3}\frac{\partial F}{\partial z_J'}z_J'\right)\delta M + \sum_{J=1}^{3}\frac{\partial F}{\partial z_J'}\delta z_J\right]_i^f = 0 \quad [28]$$

which is to be identically satisfied for all systems of variations $(\delta M, \delta z_J)$ consistent with the prescribed end conditions. In view of the fact that

$$\delta M_i = \delta\eta_i = \delta\epsilon_i = \delta\eta_f = \delta\epsilon_f = 0 \quad [29]$$

$$\frac{\partial F}{\partial\alpha'} = 0 \quad [30]$$

the transversality condition reduces to

$$\left(F - \frac{\partial F}{\partial\eta'}\eta' - \frac{\partial F}{\partial\epsilon'}\epsilon'\right)_f \delta M_f = 0 \quad [31]$$

For the particular case where the final Mach number is free of choice (δM_f = arbitrary) Equation [31] yields

$$\left(F - \frac{\partial F}{\partial\eta'}\eta' - \frac{\partial F}{\partial\epsilon'}\epsilon'\right)_f = 0 \quad [32]$$

a relation equivalent to

$$[-\varphi_1 + \lambda_2\varphi_2 - \lambda_3(\mu + \varphi_3)]_f = 0 \quad [33]$$

The Family of Variable Thrust Sub-Arcs

In this section, the variable thrust sub-arc of the extremal solution is analyzed in detail, because it plays a fundamental role in the solution of the variational problem. Since $\lambda_3 = 0$

[5] In all variational problems where one variable (μ) is bounded between a lower value $(\mu = 0)$ and an upper value (μ_{max}), sub-arcs $\mu = 0$ and sub-arcs $\mu = \mu_{max}$ must be considered in addition to those sub-arcs which are solutions of a set of Euler equations. With the device of parametric representation of the engine characteristics the sub-arcs $\mu = 0$ and $\mu = \mu_{max}$ are formally presented as Eulerian solutions. In this way Equations [23 to 25] hold for all sub-arcs forming one extremal arc.

[6] To prescribe the initial and final values of η and ϵ implies to specify the initial and final values of the mass m, in view of Equations [6 and 8].

at points of such a sub-arc, the Euler Equation [23] leads to

$$\lambda_2 = \text{Const} \quad \ldots [34]$$

After introducing the new constant

$$C = -\frac{1}{\lambda_2} \quad \ldots [35]$$

and the definitions

$$\varphi = \varphi_2 + C\varphi_1 \quad \ldots [36]$$

$$\psi = \psi_2 + C\psi_1 \quad \ldots [37]$$

the Euler Equation [24] is rewritten as

$$\left(\frac{\partial \psi}{\partial M}\right)_\epsilon - \left(\frac{\partial \varphi}{\partial \epsilon}\right)_M = 0 \quad \ldots [38]$$

where the subscripts to parentheses denote quantities to be kept constant when calculating partial derivatives.[7]

Further Transformation of Coordinates

It is of interest to notice that Equation [38] can also assume the elegant form

$$\left(\frac{\partial \varphi}{\partial M}\right)_{\epsilon_*} = 0 \quad \ldots [39]$$

following a transformation of coordinates from the (ϵ, M) plane into the (ϵ_*, M) plane, where ϵ_* is a new non-dimensional variable, defined as

$$\epsilon_* = \epsilon \exp\left(\frac{M}{M_e}\right) \quad \ldots [40]$$

Equation [39] suggests an immediate graphical procedure for finding the points of the variable thrust sub-arc: It consists of determining the Mach number M for which φ is stationary, for given values of C and ϵ_*.

Explicit Solutions

The relationship between Mach number M and non-dimensional mass ϵ can be explicitly solved in terms of the latter variable if Equations [13 to 16, 36 and 37] are introduced into Equation [38]. The following result is obtained

$$\epsilon = M \frac{C_{DO}M\left(1 + \frac{M}{M_e}\right) + M^2\frac{dC_{DO}}{dM} + C\left[C_{DO}\left(2 + \frac{M}{M_e}\right) + M\frac{dC_{DO}}{dM}\right]}{1 + C\gamma M_e} \quad \ldots [41]$$

For the all supersonic domain the relationship between zero-lift drag coefficient and Mach number may be represented in the form

$$C_{DO}M^2 = yM^x \quad \ldots [42]$$

where y and x are appropriate constants.[8] In such a case Equation [41] is rewritten as

$$\epsilon = yM^x \frac{x - 1 + \frac{M}{M_e} + C\left[\frac{x}{M} + \frac{1}{M_e}\right]}{1 + C\gamma M_e} \quad \ldots [43]$$

From Equations [43 and 11], the derivative of the non-dimensional altitude η with respect to the Mach number (M) may be calculated. After extensive manipulations, one obtains

$$\frac{d\eta}{dM} = \frac{M^2 + 2xM_eM + x(x-1)M_e^2}{\gamma M_eM^2 + [\gamma M_e^2(x-1) - 1]M - xM_e} \quad \ldots [44]$$

Analogously, the derivative of the non-dimensional time (τ) with respect to the Mach number (M) is supplied by

$$\frac{d\tau}{dM} = \frac{M^2 + 2xM_eM + x(x-1)M_e^2}{\gamma M_eM^2 + [\gamma M_e^2(x-1) - 1]M - xM_e}\frac{1}{M} \quad \ldots [45]$$

Equations [44 and 45] can then be integrated yielding[9]

$$\eta = A(M) + C_1 \quad \ldots [46]$$

$$\tau = B(M) + C_2 \quad \ldots [47]$$

where C_1 and C_2 are integration constants, chosen so that $A(M) = B(M) = 0$ at $M = 1$. The functions $A(M)$ and $B(M)$ are defined as

$$A(M) = \frac{M-1}{\gamma M_e} + \frac{K_1K_3 - K_4}{K_1 - K_2}\log\frac{M - K_1}{1 - K_1} + \frac{K_4 - K_2K_3}{K_1 - K_2}\log\frac{M - K_2}{1 - K_2} \quad \ldots [48]$$

$$B(M) = \frac{K_1K_5 - K_6}{K_1 - K_2}\log\frac{M - K_1}{1 - K_1} + \frac{K_6 - K_2K_5}{K_1 - K_2}\log\frac{M - K_2}{1 - K_2} + K_7\log M \quad \ldots [49]$$

where

$$K_1 = \frac{1 - \gamma(x-1)M_e^2 + \sqrt{[1 - \gamma(x-1)M_e^2]^2 + 4\gamma xM_e^2}}{2\gamma M_e} \quad \ldots [50]$$

$$K_2 = \frac{1 - \gamma(x-1)M_e^2 - \sqrt{[1 - \gamma(x-1)M_e^2]^2 + 4\gamma xM_e^2}}{2\gamma M_e} \quad \ldots [51]$$

$$K_3 = \frac{1}{\gamma M_e}\left[\frac{1}{\gamma M_e} + M_e(x+1)\right] \quad \ldots [52]$$

$$K_4 = -\frac{x}{\gamma M_e}\left[\frac{1}{\gamma} + (x-1)M_e^2\right] \quad \ldots [53]$$

$$K_5 = (x-1)M_e + \frac{1}{\gamma M_e} \quad \ldots [54]$$

$$K_6 = -[M_e(x-1)]^2 - \frac{x+1}{\gamma} \quad \ldots [55]$$

$$K_7 = M_e(1-x) \quad \ldots [56]$$

Remarks

For given values of x, y and M_e, Equation [43] has the form $\epsilon = \epsilon(M, C)$ and therefore embodies a single infinity of curves, one for each value of the constant C. After simple transformations, Equation [43] can also be written as

$$\frac{\epsilon}{yM^x} = x - 1 + \frac{M}{M_e} - \frac{C\gamma}{M(1 + C\gamma M_e)}(M - K_1)(M - K_2) \quad \ldots [57]$$

With reference to the positive quadrant of the (ϵ, M) plane, all curves of the family $\epsilon = \epsilon(M, C)$ have a common point, the one of coordinates

[7] It must be emphasized that there is *one* variable thrust sub-arc for *each* value of the constant C. The latter must be determined so as to satisfy the end conditions of the variational problem.

[8] In practice, x is a number bounded between 1 and 2.

[9] By simple transformations Equations [46 and 47] can be reconduced to a form similar to Equations [61 and 62] of (9).

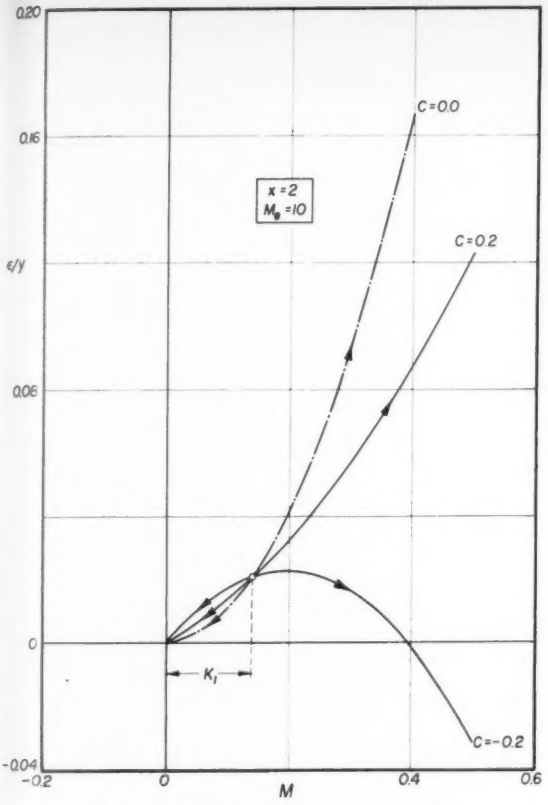

Fig. 2 Family of variable thrust sub-arcs

$$\overline{M} = K_1 \quad \ldots\ldots [58]$$

$$\bar{\epsilon} = yK_1^{x}\left(x - 1 + \frac{K_1}{M_e}\right) \quad \ldots\ldots [59]$$

Its significance can be understood by calculating the instantaneous acceleration along the variable thrust sub-arc

$$\frac{\dot{V}}{g \sin \theta} = \frac{dM}{d\tau} = \gamma M_e M \frac{(M - K_1)(M - K_2)}{M^2 + 2xM_eM + M_e^2x(x-1)} \quad \ldots\ldots [60]$$

and by showing that $\dot{V}$ is negative for $M < K_1$, zero for $M = K_1$, and positive for $M > K_1$. As a conclusion, after entering the variable thrust sub-arc, the missile moves so as never to cross the line $M = K_1$ of the (ϵ, M) plane. Since in practice K_1 is always less than unity, it seems logical to conclude that as far as the variable thrust sub-arc is concerned the region of the (ϵ, M) plane where $M < K_1$ has little or no interest for flight operations.

For the particular case where $x = 2$, $M_e = 10$, Equation [43] is plotted in Figs. 2 and 3. In Fig. 2, the neighborhood of the point of coordinates $(\bar{\epsilon}, \overline{M})$ is indicated. In Fig. 3, the region $M > K_1$ is represented. As the analysis of (13) shows, all the solid lines of Fig. 3 are associated with minimum time solutions, while all dotted lines of Fig. 3 yield maximum time solutions. The special sub-arc $C = 0$ is the transition line from minimum time solutions to maximum time solutions.

Constant Thrust Sub-Arcs

For the pieces of extremal solution which must be flown either with maximum engine output ($\mu = \mu_{max}$) or by coasting ($\mu = 0$), the equations of motion cannot be integrated, in general, into a closed form. Approximate procedures are usually in order, except in a few cases whose detailed discussion can be found in (13).

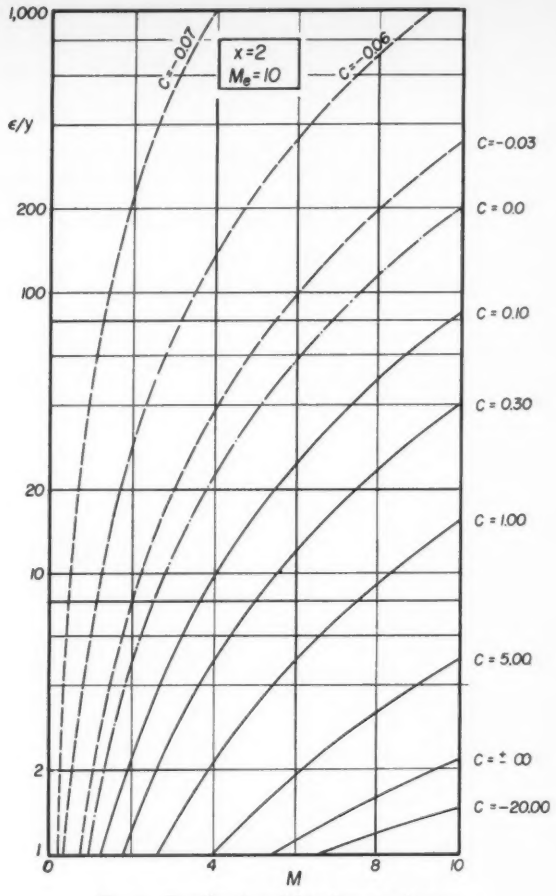

Fig. 3 Family of variable thrust sub-arcs

Solution of the Boundary Value Problem for Fixed Final Mach Number

In the previous sections general solutions have been derived for the problem of extremizing τ_f. It has been shown that the extremal arc is composed of sub-arcs $\mu = 0$, sub-arcs $\mu = \mu_{max}$ and sub-arcs to be flown with regulated thrust.

Here the boundary value problem is investigated. It consists of determining that combination of sub-arcs which satisfies a set of prescribed end conditions. Assuming that the initial Mach number is $M_i = 0$, the extremal path generally starts (11) with an initial sub-arc IA flown with maximum engine output (Fig. 4), continues with an intermediate sub-arc AB flown with variable thrust, and terminates with a final sub-arc BF. The latter can either be traveled with a coasting flight (Case I) or with maximum engine output (Case II), depending upon the boundary conditions of the problem (Fig. 4). It is to be noted that, for given initial and final values of Mach number, altitude and mass, the end points I and F are completely defined in the (M, η, ϵ) space. The problem is, therefore, one with fixed end points. The six coordinates of corner points A and B are unknown and must be determined.

Determination of Corner Points in the (ϵ, M) Plane

The first step is to locate the end points I and F of the desired extremal solution in the (ϵ, M) plane (Fig. 5). Assuming that the initial Mach number is $M_i = 0$, the sub-arc $\mu = \mu_{max}$ starting at I is calculated by forward integration of the equations of motion. Analogously, the two possible sub-

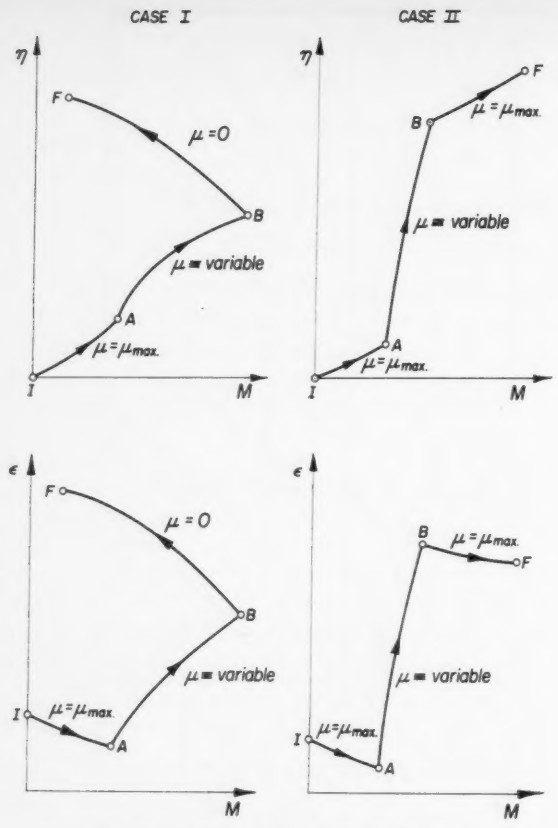

Fig. 4 Possible combinations of sub-arcs

arcs $\mu = 0$ and $\mu = \mu_{max}$ arriving at F are calculated by backward integration of the equations of motion. In this way several possible couples of corner points A and B are individualized, with a graphical procedure, in the (ϵ, M) plane. Notice that there is one couple of corner points A and B for each value of the constant C characterizing the variable thrust sub-arc of the extremal solution (Fig. 5).

The Λ-Function

The next step is to account for the altitude condition. The below-indicated functional is defined

$$\Lambda = \eta_f - \eta_i + \int_{IABF} (\varphi_2 + \psi_2\epsilon')dM \quad \ldots\ldots [61]$$

The latter has the form $\Lambda = \Lambda(C)$, if it is applied to the family of trajectories $IABF$ of Fig. 5.

The final step is to search, among all trajectories $IABF$ of Fig. 5, that special path such that $\Lambda(C) = 0$. This trajectory evidently satisfies all boundary conditions of the variational problem.

Numerical Example

To illustrate the theory, a numerical example is carried out for the set of conditions[10]

$$\begin{array}{llll} M_i = 0 & \eta_i = 0 & \epsilon_i & = 0.9 \\ M_f = 0 & \eta_f = 25 & m_p/m_i & = 0.6. \\ M_e = 10 & x = 2 & \mu_{max} & = 1.35 \\ y = 0.3 & & & \end{array} \quad \ldots\ldots [62]$$

[10] The symbol m_p stands for propellant mass; m_i denotes initial or take-off mass.

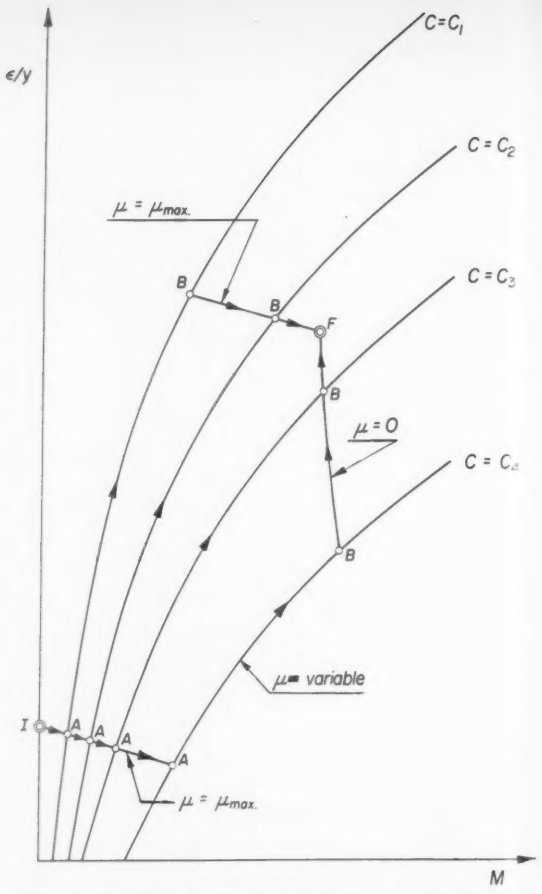

Fig. 5 Combination of sub-arcs in the $(\epsilon/y, M)$ plane

After determining all possible couples of corner points A and B in the (ϵ, M) plane, the Λ-function is calculated and plotted in Fig. 6. Since the extremal trajectory is to be such that $\Lambda(C) = 0$, a fundamental result is obtained: For a given set of end conditions there are two extremizing trajectories, the path $IABF$ associated with the variable thrust sub-arc $C = -0.06$, and the path $IABF$ associated with the variable thrust sub-arc $C = 0.68$.

Both solutions $IABF$ include an initial sub-arc IA flown with maximum thrust, an intermediate sub-arc AB flown with variable thrust and a final sub-arc BF flown by coasting (Figs. 7 and 8). As shown in (13) the trajectory $IABF$ associated with $C = -0.06$ maximizes the time ($\tau_f = 8.82$); the trajectory $IABF$ associated with $C = +0.68$ is the brachistochronic one, i.e., the path of minimum time ($\tau_f = 7.29$).

Solution of the Boundary Value Problem for Free Final Mach Number

If the final Mach number M_f is free of choice, two possible techniques may be used in solving the boundary value problem.

1 One method is based on the transversality condition of the Calculus of Variations, represented by Equation [33]. This technique requires that the distribution of Lagrange multipliers be calculated at points of the sub-arcs $\mu = \mu_{max}$ and $\mu = 0$. For the sub-arcs $\mu = 0$ the computation of λ_2 and λ_3 is materially simplified (13) since the Euler Equation [23] admits a first integral

$$\lambda_2 - \lambda_3 \frac{\varphi_3^2}{\psi_3} = C_3 \quad \ldots\ldots [63]$$

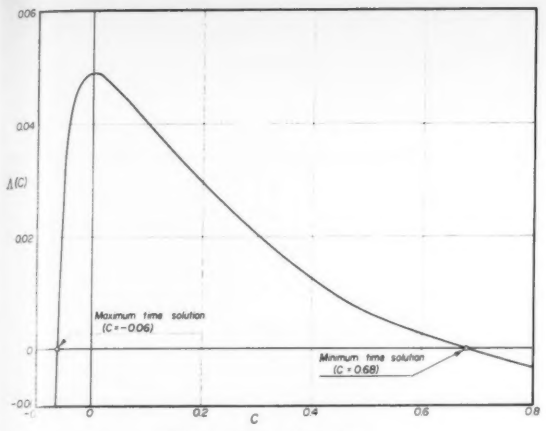

Fig. 6 The Δ-function

C_3 being an integration constant. The calculation of λ_2 and λ_3 is ultimately reconduced (13) to a process of simple quadratures.

2 Another method of solution consists in attacking the fixed end-points problem for several values of the final Mach numbers M_f; calculating the time τ_f necessary to reach each final Mach number M_f; plotting the resulting function $\tau_f(M_f)$; and finding, by a graphical procedure, the trajectory yielding $(\tau_f)_{min}$. This method bypasses the natural condition resulting into Equation [33], but it is slightly less precise in the determination of the optimum trajectory.

3 The above two methods (which yield consistent results) have been developed to a considerable extent in (13). As shown by a typical example, the computational effort involved is approximately the same with both techniques. Method 2 is closer perhaps to the immediate intuition of the engineer and offers a clear picture of the effect which a deviation from the optimum flight program has on the final time τ_f.

Numerical Example

A sample calculation is now carried out for the following set of conditions[11]

$$\begin{array}{llll} M_i = 0 & \eta_i = 0 & \epsilon_i = 0.9 \\ M_f = 10 & \eta_f = 2 & \epsilon_f = 5.624 \\ & x = 2 & \\ m_p/m_i = 0.62 & y = 0.3 & \mu_{max} = \infty \end{array} \quad \ldots . [64]$$

For the sake of brevity only method 2 is employed, even though identical results have been obtained with method 1 in (13).

As a first step the fixed end-points problem is solved for several final Mach numbers M_f. The family of brachistochronic paths $IABF$ satisfying the above set of conditions and leading to different fixed final Mach numbers M_f is computed (Figs. 9 to 12).

Each brachistochronic path $IABF$ can be described either in terms of the final Mach number M_f or in terms of the constant C associated with the singular sub-arc AB. Notice that each trajectory $IABF$ includes: an initial pulse-burning sub-arc IA, a central sub-arc AB flown with variable thrust and a final sub-arc BF. With regard to the latter, the analysis shows:

1 For $M_f = 7.7$ ($C = 0.36$) the final point F is reached with regulated thrust (Figs. 9 to 12).

[11] Notice that to simplify the analysis the ideal case of an engine capable of a pulse burning [$\mu_{max} = \infty$] is considered.

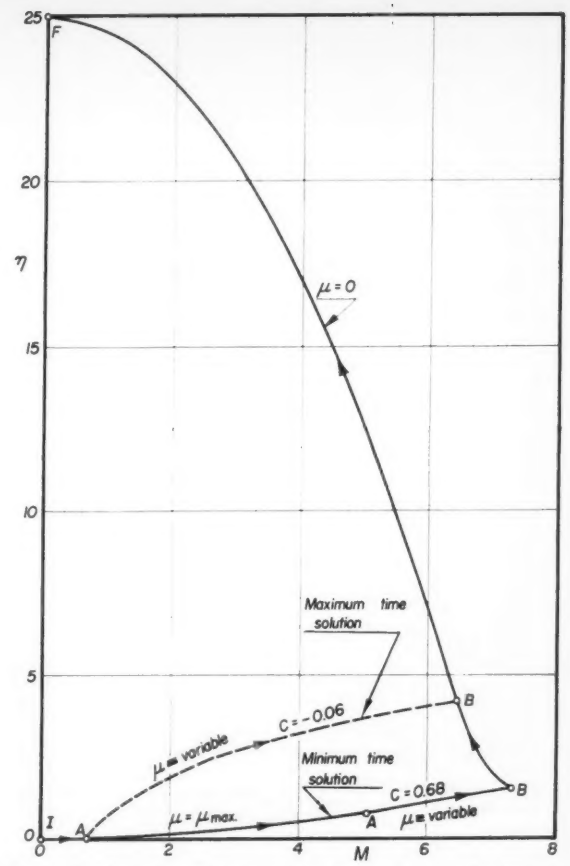

Fig. 7 Particular brachistochronic trajectory for fixed final Mach number

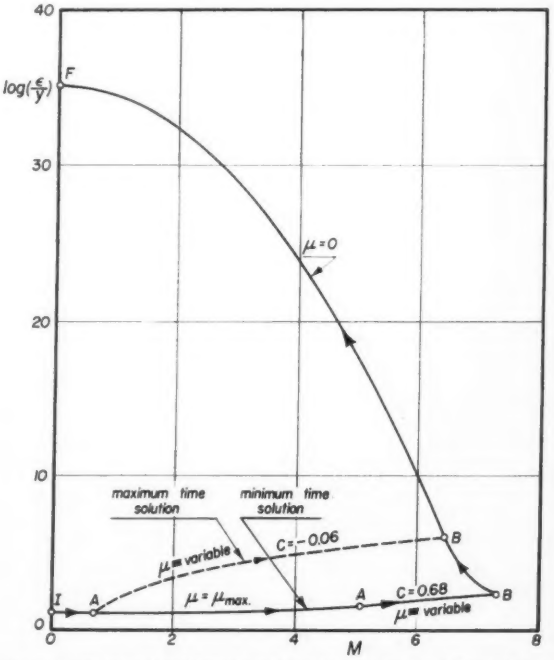

Fig. 8 Particular brachistochronic trajectory for fixed final Mach number

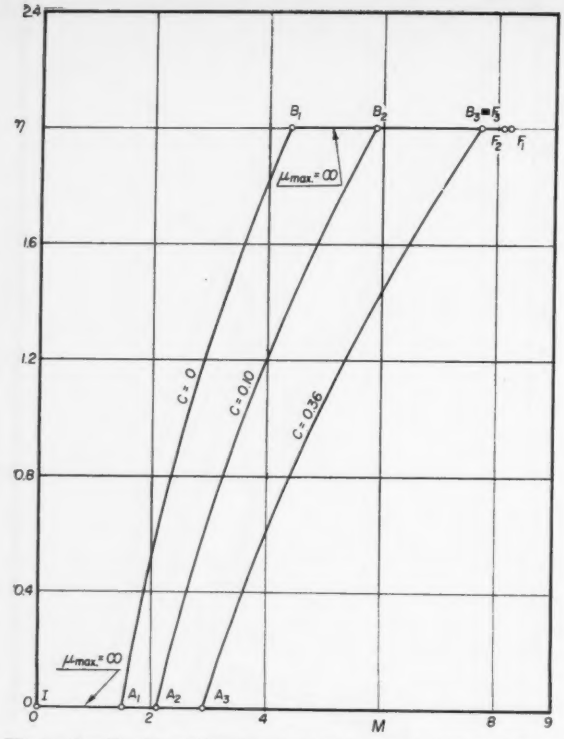

Fig. 9 Family of brachistochronic paths leading to different fixed final Mach numbers

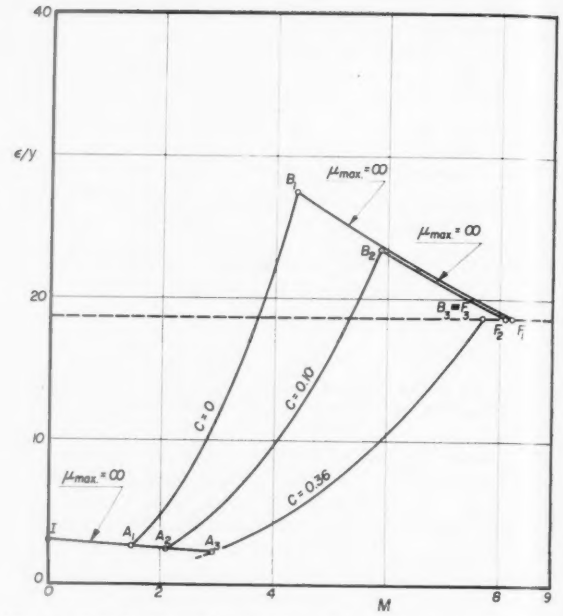

Fig. 10 Family of brachistochronic paths leading to different fixed final Mach numbers

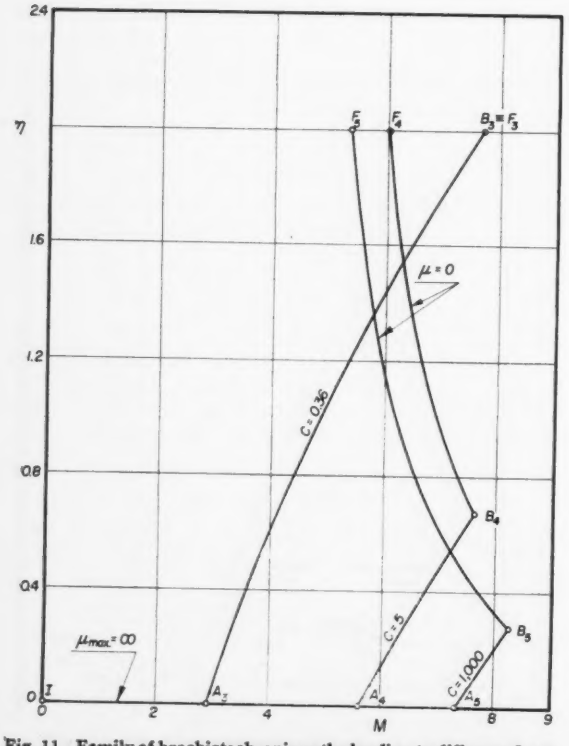

Fig. 11 Family of brachistochronic paths leading to different fixed final Mach numbers

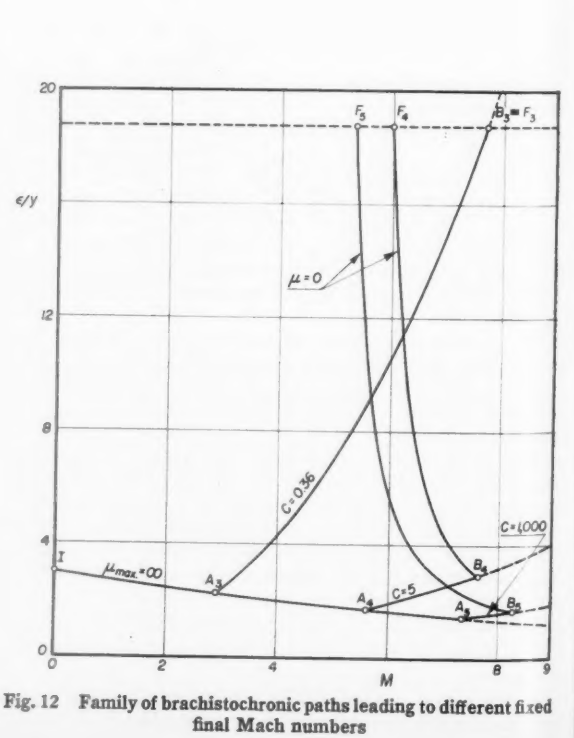

Fig. 12 Family of brachistochronic paths leading to different fixed final Mach numbers

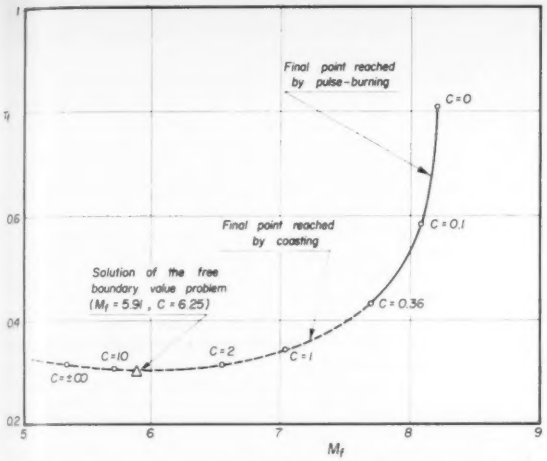

Fig. 13 Time necessary to reach the final point along a brachistochronic path as a function of the final Mach number

2 For $M_f > 7.7$ ($C < 0.36$) the final point F is reached with a pulse-burning sub-arc BF (Figs. 9 and 10).

3 For $M_f < 7.7$ ($C > 0.36$) the final point F is reached with a coasting sub-arc BF (Figs. 11 and 12).

Once all possible brachistochronic paths $IABF$ are drawn in the (η, M) plane the time necessary to reach the final point is computed from the line integral (Fig. 13)

$$\tau_f = \int_{\eta_i}^{\eta_f} \frac{d\eta}{M} = \tau_f(M_f) \quad \ldots\ldots\ldots\ldots [65]$$

Clearly the special path minimizing the time for free final Mach number is to be consistent with the necessary condition

$$\frac{d\tau_f}{dM_f} = 0 \quad \ldots\ldots\ldots\ldots [66]$$

As it is evident from the integral [65] the condition [66] cannot be met by any of the paths $IABF$ ending with a pulse-burning (Fig. 9). This equation, on the other hand, can be met by a path $IABF$ ending with a coasting flight (Fig. 11). In this connection, Fig. 13 shows that the trajectory solving the free boundary value problem is the path $IABF$ corresponding to $C = 6.25$, $M_f = 5.91$ and terminating with a coasting flight. Such a path is indicated in Figs. 14 and 15.

Remarks

For the particular examples previously discussed the extremal path was shown to be composed of sub-arcs of three different kinds. There are cases, however, where the composition of the extremal path is quite different from the one indicated in Figs. 6 to 15. For instance, the variable thrust sub-arc may not be present in the solution; or the extremal arc may reduce to a single sub-arc flown with maximum engine output. The particular extremal arc associated with a given problem strongly depends on the characteristics of the airframe and of the engine and, in addition, on the boundary conditions of the problem.

Conclusions

The brachistochronic burning program for a rocket powered missile moving along a rectilinear, near vertical trajectory imbedded in an isothermal medium is analyzed. Under suitable assumptions for the drag function, solutions in a closed form are obtained for the relationships between altitude, time, mass, and Mach number.

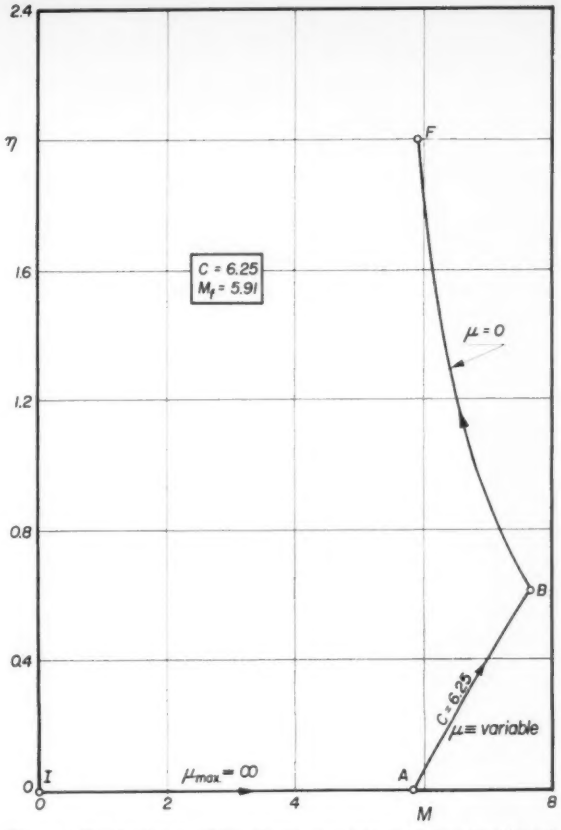

Fig. 14 Particular brachistochronic trajectory for free final Mach number

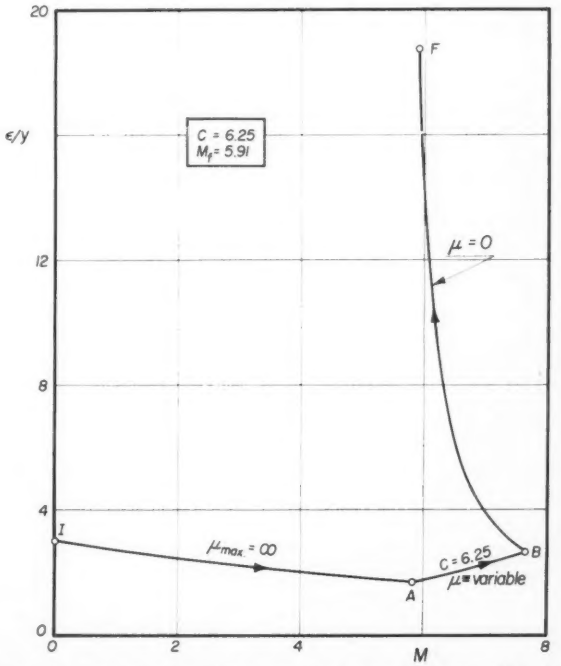

Fig. 15 Particular brachistochronic trajectory for free final Mach number

The boundary value problem is investigated for the case where the final Mach number is prescribed and the case where it is free of choice.

For the fixed end-points problem, a numerical example shows that a given set of end conditions may be met by *two* different extremizing solutions: The brachistochronic solution and one which maximizes the time.

In regard to the free boundary value problem, two procedures are developed. One is based on the transversality condition; the other one is conceived as a sequence of repeated solutions of the boundary value problem for several fixed final Mach numbers. The amount of computational work necessary to predict the optimum path is roughly the same in both methods.

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Technical Notes

A Method for Calculating Impact Points of Ballistic Rockets: Convenient Representations

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Nomenclature

a = semimajor axis of orbit ellipse ($a = [R_A + R_P]/2$), m

A_L' = fraction of the area of the ellipse swept out by r in going from the perigee to R_L ($A_L = [\cos^{-1}\ [(Y_L^2 - 1)/\epsilon] - \epsilon(1 - [1 - Y_L^2]^2/\epsilon^2)^{1/2}]/2\pi$)

α = angle in the inertial frame between north and the projection of the launching direction in the inertial frame on the horizontal plane measured clockwise as seen from above, deg or rad

α' = angle in the Earth's frame between north and the projection of the launching direction on the horizontal plane measured clockwise as seen from above, deg or rad

ΔL = difference in longitude between L_I and L_L ($\Delta L = L_I - L_L$), deg or rad

$\Delta L'$ = change in longitude in the Earth's frame, deg or rad

$\Delta\phi$ = total angle r has turned through from $r = R_L$ on one side to $r = R_L$ on the other side of the ellipse in the flight of the rocket, deg or rad

ϵ = eccentricity of the orbit ellipse ($\epsilon = (1 - Y_L^2 \cos^2\theta\ [2 - Y_L^2])^{1/2}$)

G = Newton's gravitational constant ($G = 6.670 \times 10^{-11}$ nt $\cdot$ m$^2 \cdot$ kg^{-2})

h = peak altitude reached by the rocket above R_L, m

θ = angle between the launching direction and the horizontal in the inertial frame, deg or rad

θ' = angle between the launching direction and the horizontal in the Earth's frame, deg or rad

L_I = longitude of the point of impact at a distance R_L from the center of the Earth, deg or rad

L_L = longitude of the Earth where rocket was launched, deg or rad

λ_I = latitude of the point of impact at a distance R_L from the center of the Earth, deg or rad

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EDITOR'S NOTE: The Technical Notes and Technical Comments sections of JET PROPULSION are open to short manuscripts describing new developments or offering comments on papers previously published. Such manuscripts are published without editorial review, usually within two months of the date of receipt. Requirements as to style are the same as for regular contributions (see masthead page).

λ_L = latitude of the Earth where rocket was launched, deg or rad
λ_I' = impact latitude in the Earth's frame, deg or rad
M_E = mass of the Earth ($M_E = 5.983 \times 10^{24}$ kg)
P_I = impact point
P_L = launching point
P_I' = impact point in the Earth's frame
ϕ_L = the value of the polar angle (i.e., geocentric angle) of the orbit measured from the perigee to the launching point, deg or rad
R_A = apogee distance, m
R_L = distance of launching point from center of Earth, m
R_P = perigee distance, m
t = total time of flight of the rocket above R_L, sec
t_c = period of a circular orbit of radius R_L ($t_c = 2\pi R_L/V_c$), sec
T_e = period of the rocket in the complete ellipse ($T_e = 2\pi (GM_E)^{-1/2}a^{3/2}$), sec
V_c = velocity of an object in a circular orbit of radius R_L ($V_c = [GM_E/R_L]^{1/2}$), m · sec^{-1}
V_L = launching velocity magnitude in the inertial frame, m · sec^{-1}
V_L' = launching velocity magnitude in the Earth's frame, m · sec^{-1}
$\vec{V}_L'$ = launching velocity vector of the rocket in a reference frame fixed with respect to the Earth, m · sec^{-1}
ω = angular rotation of the earth ($\omega = 7.272 \times 10^{-5}$), rad · sec^{-1}
Y_L = dimensionless launching velocity in the inertial frame ($Y_L = V_L/V_c$)

Introduction

A METHOD for determining impact points of ballistic rockets on a rotating Earth has been developed by Singer and Wentworth (1).[3] In the present note, a dimensionless representation is given which puts the results of (1) in a more convenient form. Also, (1) is amended to include nonequatorial launchings.

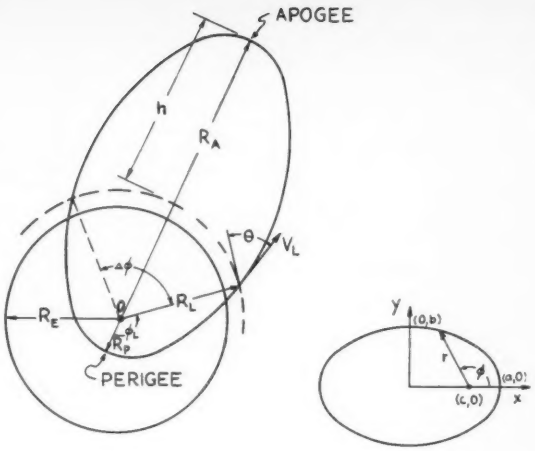
The method used in (1) to calculate the impact point of a ballistic rocket considers the elliptic trajectory of the coasting rocket in an inertial reference frame fixed with respect to the stars rather than using a coordinate system which rotates with the Earth. The rotation of the Earth is then taken into account by adding a suitable horizontal component to the launching velocity of the rocket, and by considering the angle turned through by the earth during the time of flight of the rocket.

The launching parameters are the velocity $\vec{V}_L'$ of the rocket just after final burnout (the subsequent flight of the rocket being therefore unpowered) and the latitude λ_L, longitude L_L, and radius from the center of the Earth R_L where final burnout occurs. For simplicity, the impact point is taken to be the same distance R_L from the center of the Earth, and it is also assumed that the final burnout takes place above the atmosphere, so air friction can be neglected.

Specifically, the launching velocity as seen from the Earth's frame $\vec{V}_L'$ is designated by its magnitude V_L', the angle it makes with the horizontal θ', and the angle its projection makes with north, measured clockwise α'. Similar quantities as seen from the nonrotating inertial frame, V_L, θ, and α, are calculated from these, and V_L and θ directly determine the orbit ellipse. The orbit ellipse in turn determines the angular range $\Delta\phi$, the peak altitude h, and the time of flight t of the coasting rocket. Knowledge of these quantities coupled with the launching latitude λ_L and longitude L_L then enables one finally to determine, with the aid of some spherical trigonometry, the point of impact.

However, since the publication of (1), it has been found possible to put the important results of that paper into convenient dimensionless form. Thus, the angular range $\Delta\phi$, the dimensionless time of flight of the rocket t/t_c, and the dimensionless altitude h/R_L are expressed as functions of the dimensionless launching velocity Y_L and launching angle θ, in the first part of this note.

Also, the authors of (1) received a communication from Professor E. E. Zukoski[4] who pointed out that Equations [19 and 20] of that paper were in error for nonequatorial launchings and indicated (2) as containing the correct solution. Therefore, the second part of this note is devoted to a derivation of the correct forms of those equations which give the impact point as a function of the launching point and the launching parameters.



$V_c \equiv (GM_E/R_L)^{1/2}$

$y_L \equiv V_L/V_c$

$\epsilon = (1 - y_L^2 \cos^2\theta[2 - y_L^2])^{1/2}$

$\Delta\phi = 2\sin^{-1}(y_L^2 \sin 2\theta/2\epsilon)$

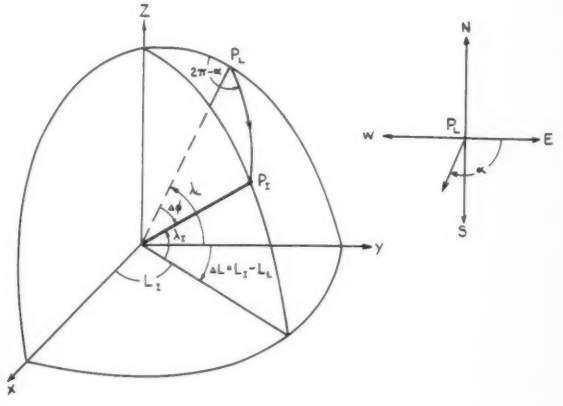
$h/R_L = -1 + (1+\epsilon)/(2 - y_L^2)$

$(x^2/a^2) + (y^2/b^2) = 1$

$r = a(1-\epsilon^2)/(1+\epsilon\cos\phi)$

$\epsilon = (1 - b^2/a^2)^{1/2}$

Fig. 1 The elliptic orbit of a ballistic rocket showing notation used



$\lambda_I = \sin^{-1}(\cos\Delta\phi \sin\lambda_L + \sin\Delta\phi \cos\alpha \cos\lambda_L)$

$L_I - L_L \equiv \Delta L = \cot^{-1}(-\sin\lambda_L \cot\alpha + \cos\lambda_L \cot\Delta\phi / \sin\alpha)$

Fig. 2 The angular trajectory $\Delta\phi$ of a ballistic rocket on the surface of the Earth showing notation used

[3] Numbers in parentheses indicate References at end of paper.

[4] Assistant Professor of Jet Propulsion, California Institute of Technology, Pasadena, Calif.

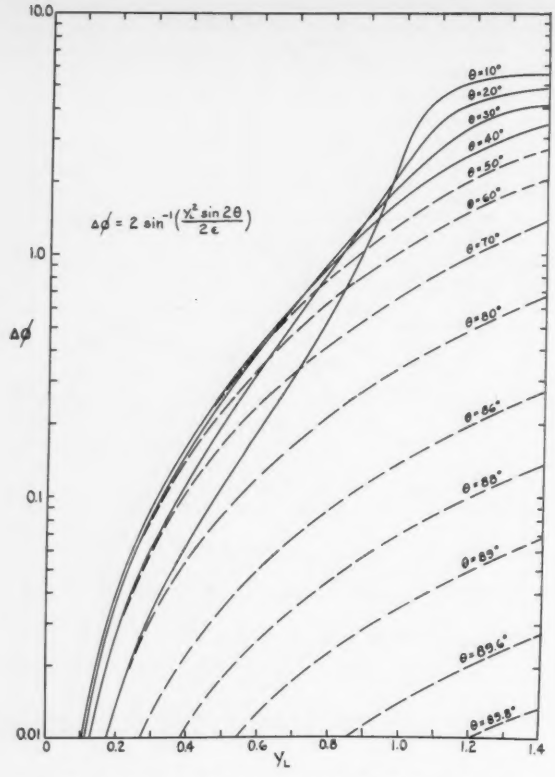


Fig. 3 The angular range $\Delta\phi$ of a ballistic rocket

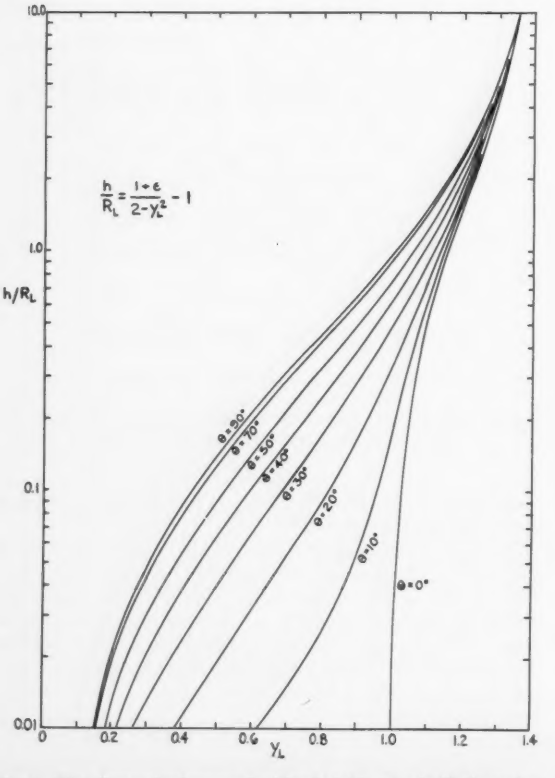


Fig. 4 The dimensionless peak altitude h/R_L of a ballistic rocket

Dimensionless Representation

It is most convenient to work with dimensionless quantities in the following expressions, and with that in mind we will make use of the so-called circular velocity V_c, that is, the velocity required for an object to maintain a circular orbit at a distance R_L from the center of the Earth. Then the dimensionless launching velocity Y_L is the ratio of the actual launching velocity V_L at the launching radius R_L to the circular velocity at that radius.

$$Y_L \equiv V_L/V_c \qquad V_c \equiv (GM_E/R_L)^{1/2} \quad \ldots\ldots [1]$$

Using the definition of V_c, Equations [6] of (1) become

$$\begin{aligned} A &= 2V_c^2 - V_L^2 \\ B &= 2R_L V_c^2 \\ C &= R_L^2 V_L^2 \cos^2\theta \end{aligned} \quad \ldots\ldots [2]$$

Substituting [2] into Equations [8] of (1), we have

$$\begin{aligned} R_A &= R_L V_c^2 (1 + [1 - (2V_c^2 - V_L^2)(V_L^2\cos^2\theta)/V_c^4]^{1/2})/(2V_c^2 - V_L^2) \\ R_P &= R_L(1 - [1 - (2 - Y_L^2)(Y_L^2\cos^2\theta)]^{1/2})/(2 - Y_L^2) \end{aligned} \quad \ldots\ldots [3]$$

Then making use of Equations [1, 2, and 3], Equations [10, 11, 15 and 17] of (1) reduce, with a few minor changes in notation, to

$$\begin{aligned} \epsilon &= (1 - Y_L^2\cos^2\theta[2 - Y_L^2])^{1/2} \\ \phi_L &= \cos^{-1}([Y_L^2\cos^2\theta - 1]/\epsilon) \\ a &= R_L/(2 - Y_L^2) \\ 2\pi A_L &= \cos^{-1}[(Y_L^2 - 1)/\epsilon] - \epsilon[1 - (1 - Y_L^2)^2/\epsilon^2]^{1/2} \end{aligned} \quad \ldots\ldots [4]$$

Using [4], Equation [18] of (1) becomes

$$\Delta\phi = 2\sin^{-1}(Y_L^2[\sin 2\theta]/2\epsilon) \quad \ldots\ldots [5]$$

The dimensionless peak altitude h/R_L is

$$h/R_L = (R_A - R_L)/R_L = -1 + (1+\epsilon)/(2 - Y_L^2) \quad \ldots\ldots [6]$$

Finally, defining the circular period t_c of an object in a circular orbit of radius R_L about the Earth by

$$t_c \equiv 2\pi R_L/V_c \quad \ldots\ldots [7]$$

Equation [12] of (1) may be written

$$T_e = t_c(a/R_L)^{3/2} \quad \ldots\ldots [8]$$

so that we have for the dimensionless time of flight, making use of Equation [13] of (1)

$$t/t_c = [1 - 2A_L]T_e/t_c = [1 - 2A_L]/(2 - Y_L^2)^{3/2} \quad \ldots\ldots [9]$$

Equations [5, 6 and 9] are plotted as functions of Y_L with θ as a parameter in Figs. 3, 4 and 5. These graphs then easily permit the determination of the angular range (in radians) $\Delta\phi$, the altitude h and the time of flight t as functions of the launching velocity V_L and angle θ.

Impact Point

Suppose that the launching velocity vector as seen from the earth $\vec{V}_L'$ is defined by its magnitude V_L', the angle it makes with the horizontal plane θ' and the angle measured clockwise that its projection on the horizontal plane makes with north α' (see Fig. 2).

To determine the corresponding quantities in a fixed reference frame, one must add to $\vec{V}_L'$ the velocity to the east due to the rotation of the Earth $\omega R_L \cos\lambda_L$. Then a consideration of the geometry of the situation yields

$$\begin{aligned} \alpha &= \tan^{-1}[(V_L'\sin\alpha'\cos\theta' + \omega R_L\cos\lambda_L)/V_L'\cos\theta'\cos\alpha'] \\ &= \tan^{-1}[\tan\alpha' + \omega R_L\cos\lambda_L/V_L'\cos\theta'\cos\alpha'] \end{aligned} \quad \ldots\ldots [10]$$

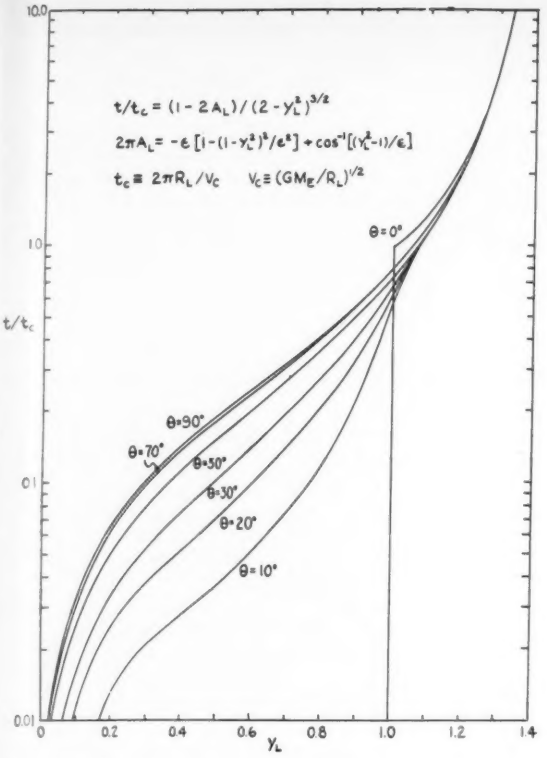


Fig. 5 **The dimensionless time of flight t/t_c of a ballistic rocket**

Also

$$\tan\theta = V_L'\sin\theta'/\eta \qquad \eta = V_L'\cos\theta'\cos\alpha'/\cos\alpha \ldots [11]$$

or

$$\theta = \tan^{-1}[\tan\theta'\cos\alpha/\cos\alpha'] \ldots [12]$$

and

$$V_L = V_L'\sin\theta'/\sin\theta \ldots [13]$$

In the special case of vertical launching in the Earth's frame

$$\alpha = +90 \text{ deg}$$
$$\theta = \tan^{-1}[V_L'/\omega R_L\cos\lambda_L]$$
$$\theta' = +90 \text{ deg}$$
$$V_L = [V_L'^2 + \omega^2 R_L^2\cos^2\lambda_L]^{1/2} \ldots [14]$$

It has been shown how $\Delta\phi$ and t may be determined with the aid of [12 and 13] and Figs. 3 and 5. As can be seen from Fig. 2, the dependence of λ_I and L_I on $\Delta\phi$, α, and λ_L is a fairly complicated problem in spherical trigonometry. However, it has been solved in (2), from which it follows, with appropriate changes in notation, that

$$\lambda_I = \sin^{-1}(\cos\Delta\phi\sin\lambda_L + \sin\Delta\phi\cos\alpha\cos\lambda_L)$$
$$L_I - L_L \equiv \Delta L = \cot^{-1}(-\sin\lambda_L\cot\alpha + \cos\lambda_L\cot\Delta\phi/\sin\alpha) \ldots [15]$$

In [15], the change in longitude ΔL between the launching longitude L_L and the impact longitude L_I is positive if the impact point P_I is to the east of the launching point P_L. Also, λ_I and λ_L are positive if they are in the northern hemisphere and negative if they are in the southern hemisphere.

Finally, the impact point P_I' in the Earth's reference frame is given by

$$\lambda_I' = \lambda_I \qquad \Delta L' = \Delta L - \omega t \ldots [16]$$

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2 Kooy, J. M. J. and Uytenbogaart, J. W. H., "Ballistics of the Future," McGraw-Hill, New York, 1946, pp. 157–159. *See also* Rauscher, M., "Introduction to Aeronautical Dynamics," John Wiley & Sons, New York, 1953.

Satellite Orbit Perturbations Under a Continuous Radial Thrust of Small Magnitude

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Nomenclature

F = thrust in radial direction
r = distance from the center of Earth
r_0 = radius of the initial circular orbit
R = radius of Earth
$\lambda = \dfrac{F}{mg}\left(\dfrac{r_0}{R}\right)^2$
ϕ = angular displacement

THE problem of take-off from satellite orbits has been investigated by Tsien (1),[2] Ehricke (2), Peterson (3) and others. In what follows, those investigations are supplemented by discussing the condition of continuous radial thrust when $8\lambda < 1$. Tsien's paper was based on the condition of $8\lambda > 1$.

Initially, the spaceship is in a circular orbit of radius r_0. At time $t = 0$, a radial thrust in outward direction and such that $F = \text{const}$, $m = \text{const}$ begins to act. From equations of motion

$$m(\ddot{r} - r\dot{\phi}^2) = F - mg\frac{R^2}{r^2} \ldots [1]$$

$$r^2\dot{\phi} = h = \sqrt{gR^2r_0} \ldots [2]$$

$\dot{\phi}$ is eliminated to obtain

$$\ddot{r} = \frac{F}{m} - \frac{gR^2}{r^2} + \frac{gR^2r_0}{r^3}$$

Integration and substitution of initial condition $(\dot{r})_{r=r_0} = 0$ gives

$$\dot{r}^2 = \frac{2F}{m}r_0\left(\frac{r}{r_0} - 1\right) + 2g\frac{R^2}{r_0}\left(\frac{1}{r/r_0} - 1\right) - \frac{gR^2}{r_0}\left(\frac{1}{r^2/r_0^2} - 1\right)$$

Making the substitutions

$$s = \frac{r}{r_0} \qquad P = \frac{t}{T}$$

where

$$T = \frac{2\pi r_0^{3/2}}{\sqrt{gR^2}}$$

and is the period in the original circular orbit

$$\lambda = \frac{F}{mg}\left(\frac{r_0}{R}\right)^2$$

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[2] Numbers in parentheses indicate References at end of paper.

the following results

$$\left[\frac{1}{2\pi}\frac{ds}{dP}\right]^2 = \frac{1}{s^2}\left[(s-1)(2\lambda s^2 - s + 1)\right] \ldots\ldots [3]$$

Integrating Equation [3]

$$2\pi P = \int_1^{1+b^2} \frac{sds}{\sqrt{2\lambda f(s)}} \ldots\ldots [3a]$$

where

$$f(s) = \left[s - \frac{1}{4\lambda}\left(1 - \sqrt{1-8\lambda}\right)\right]\left[s - \frac{1}{4\lambda}\left(1 + \sqrt{1-8\lambda}\right)\right](s-1)$$

The graph of $f(s)$ against s (Fig. 1) shows that s oscillates between unity and

$$\frac{1-\sqrt{1-8\lambda}}{4\lambda} = 1 + b^2$$

Performing the two transformations

$$s = 1 + z^2 \qquad z = bx \ldots\ldots [4]$$

The integral is evaluated

$$\frac{2\pi P}{\sqrt{\frac{2}{\lambda}}} = \int_0^1 \frac{(1 + b^2x^2)bdx}{\sqrt{(1-b^4x^2)(1-x^2)}} \ldots\ldots [5]$$

$$= bF(b^2, \pi/2) + \frac{1}{b}F(b^2, \pi/2) - \frac{1}{b}E(b^2, \pi/2) \ldots\ldots [6]$$

where F and E are elliptic integrals of the first and second kind, respectively.

From [2]

$$\frac{1}{T}\frac{d\phi}{dP} = \frac{h}{r^2}$$

Hence

$$\frac{d\phi}{dP} = \frac{2\pi}{s^2} \ldots\ldots [7]$$

Combining [7 and 3]

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{1}{s^2(2\lambda s^2 - s + 1)(s-1)}$$

Integration of the above between $s = 1$ and $s_{max} = 1 + b^2$ gives

$$\phi = \sqrt{\frac{2}{\lambda}}\int_0^1 \frac{bdx}{(1 + b^2x^2)\sqrt{(1-b^4x^2)(1-x^2)}} = \sqrt{\frac{2}{\lambda}}\, b\, \Pi\,(b^2, b^2, \pi/2) \ldots [8]$$

where Π is the elliptic integral of the third kind. This last expression can be shown to be equal to

$$\phi = \sqrt{\frac{2}{\lambda}}\,\frac{b}{1+b^2}\left\{F(b^2, \pi/2) + \pi/2 - F(b^2, \pi/2)\,E\left(\sqrt{1-b^4}, \theta\right) - E(b^2, \pi/2)F\left(\sqrt{1-b^4}, \theta\right) + F(b^2, \pi/2)F\left(\sqrt{1-b^4}, \theta\right)\right\}$$

where $\cot\theta = b$.

The above gives exact expressions for the perturbed orbit. However, some approximations can be executed. From Equation [6], by expanding in series

$$\frac{2\pi P}{\sqrt{2/\lambda}} = \frac{\pi}{2}b\left(1 + \frac{b^2}{2} + \frac{b^4}{4} + \ldots\right)$$

Then if $8\lambda \ll 1$, and hence, $b^2 \sim 2\lambda$, will transform the preceding equation into

$$P = \frac{1}{2}(1 + \lambda + \lambda^2 + \ldots)$$

Also Equation [8] under similar approximations will give

$$\phi = \frac{\pi}{1 + 2\lambda}$$

It is seen that a continuous radial thrust, such that $8\lambda < 1$, will cause only a temporary departure from the satellite orbit. The vehicle is bound to return, and the maximun radial displacement never surpasses in length the radius of the original circular orbit. This is seen from Fig. 2. Fig. 3 shows the perturbed orbit.

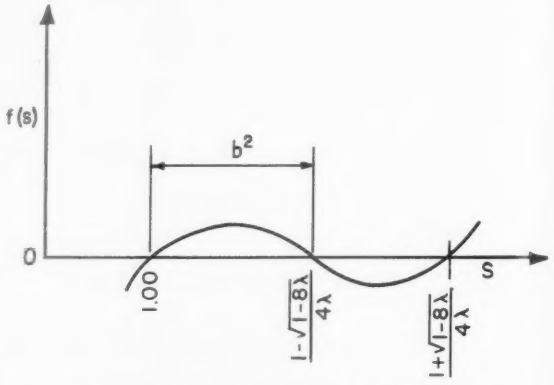

Fig. 1 The interval within which Equation [3a] can be integrated

References

1 Tsien, H. S., "Take-Off from Satellite Orbit," JET PROPULSION, vol. 23, July-August 1953, pp. 233–236.

2 Ehricke, Krafft A., "Take-Off from Satellite Orbits," JET PROPULSION vol. 23, Nov.-Dec. 1953, pp. 372–374.

3 Peterson, J. B., "Note on 'Take-Off From Satellite Orbit'," JET PROPULSION, vol. 27, Dec. 1957, pp. 1263–1264.

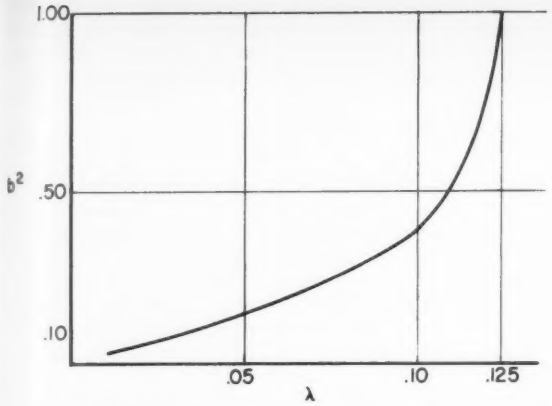

Fig. 2 The variation of b^2, the measure of the maximum radial displacement from the original circular orbit, against $\lambda = F/mg\,(r_0/R)^2$

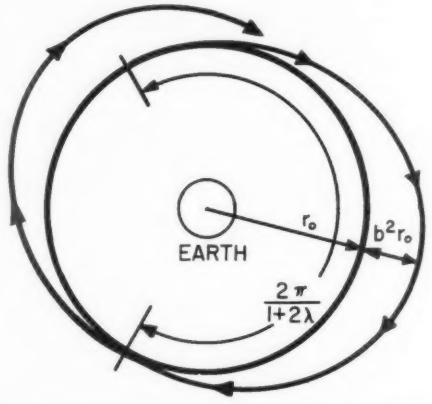

Fig. 3 Orbit perturbations under continuous radial thrust

Flat Plate Laminar Skin Friction and Heat Transfer in the Free Molecule to Continuum Flow Regimes

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SEVERAL authors have solved Rayleigh's problem (i.e., impulsive start of an infinite plate) using continuum equations with slip boundary conditions and have applied the results to estimate skin friction (1, 2)[2] and heat transfer (3) on a semi-infinite flat plate in the free molecule to continuum flow regimes. The transformation from time, in Rayleigh's problem, to distance downstream of the leading edge, in the semi-infinite flat plate problem, according to the relation $u_e t = Kx$, introduces the arbitrary constant K. This constant was chosen, in (2), such that the expression for wall shear approached the correct value in the limit of continuum flow. The resulting expression overestimated the wall shear in the free molecule regime by a factor of 2. In (1 and 3) the constant K was chosen in such a way that the wall shear and heat transfer did not agree with the correct limiting values for either the free molecule or the continuum regimes.

More recently, (4) has studied Rayleigh's problem at low Mach numbers from the point of view of kinetic theory. An approximate expression for wall shear was obtained which is valid in the entire range from free molecule to continuum flow. Surprisingly, this expression is identical—except for the numerical constants—with the expression for wall shear which results from the solution of Rayleigh's problem using continuum equations with slip boundary conditions. This suggests that the latter solution can be used as a model to provide the functional dependence of wall shear and heat transfer on Mach number and Reynolds number in the free molecule to continuum regimes. In particular, if time is replaced by distance and all numerical constants are arbitrarily modified so as to provide agreement in the limits of free molecule and continuum flow, then expressions for semi-infinite flat plate shear and heat transfer can be obtained which should give reasonable estimates of these quantities over the entire range from free molecule to continuum flow. Such an approach is outlined herein.

Consider Rayleigh's problem for a fluid with constant properties. The momentum and energy equations, with slip boundary conditions are, respectively (1,3)

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(t,0) = l\left(\frac{\partial u}{\partial y}\right)_w$$

$$u(t, \infty) = u(0, y) = u_e \qquad \text{.....[1a]}$$

$$\frac{\partial T}{\partial t} - \frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2} = 0$$

$$T(t, 0) = T_w + m\left(\frac{\partial T}{\partial y}\right)_w$$

$$T(t, \infty) = T(0, y) = T_e \qquad \text{.....[1b]}$$

where l and m are constants defining the velocity slip and temperature jump at the wall. Equation [1b] neglects dissipation. The resulting nondimensonal expressions for wall shear and heat transfer are (1,2,3)

$$C_f \equiv \frac{\left(\mu \frac{\partial u}{\partial y}\right)_w}{(\rho_e u_e^2)/2} = \frac{2\mu}{\rho_e u_e l}\,[\mathrm{erfc}(\sqrt{\nu t}/l)\exp(\sqrt{\nu t}/l)^2] \qquad \text{.....[2a]}$$

$$q_w \equiv \frac{\left(k\frac{\partial T}{\partial y}\right)_w}{\rho_e u_e C_p T_e} = \frac{\mu(T_e - T_w)}{\sigma m \rho_e u_e T_e}\left[\mathrm{erfc}\left(\sqrt{\frac{\nu t}{\sigma}}\Big/ m\right)\exp\left(\sqrt{\frac{\nu t}{\sigma}}\Big/ m\right)^2\right] \qquad \text{..[2b]}$$

Equation [2b] assumes T_w is constant. If t is replaced by x, Equations [2] have the form

$$f(x) = a\,\mathrm{erfc}\,(b\sqrt{x})\exp(b\sqrt{x})^2 \qquad \text{.....[3a]}$$

$$= a\left[1 - \frac{2}{\sqrt{\pi}}(b\sqrt{x}) + (b\sqrt{x})^2 - \ldots\right] \qquad (b\sqrt{x}) \ll 1 \qquad \text{..[3b]}$$

$$= \frac{a}{\sqrt{\pi}(b\sqrt{x})}\left[1 - \frac{1}{2(b\sqrt{x})^2} + \frac{3}{4(b\sqrt{x})^4} - \ldots\right] \qquad (b\sqrt{x}) \gg 1 \qquad \text{..[3c]}$$

where a and b are independent of x. Considering the limits of free molecule flow ($b\sqrt{x} \to 0$) and continuum flow ($b\sqrt{x} \to \infty$), a and b can be expressed as (from Equations [3b, 3c])

$$a = f(0)$$

$$b = \frac{1}{\sqrt{\pi}}\frac{f(0)}{\lim\limits_{x\to\infty}[\sqrt{x}\,f(x)]} \qquad \text{.....[4]}$$

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[1] Aeronautical Research Engineer, Lewis Flight Propulsion Laboratory.

[2] Numbers in parentheses indicate References at end of paper.

Thus a and b can be evaluated from known wall shear and heat transfer results in the free molecule and continuum regimes.

$$\frac{1}{L}\int_0^L f(x)dx = \frac{a}{(b\sqrt{L})^2}\left[\frac{2}{\sqrt{\pi}}(b\sqrt{L}) - 1 + \text{erfc}\,(b\sqrt{L})\exp(b\sqrt{L})^2\right] \quad [9]$$

Identifying $f(x)$ with C_f, we get

$$a \equiv (C_f)_{x=0} = \frac{1}{M_e}\sqrt{\frac{2}{\pi\gamma}} \quad [5a]$$

$$b \equiv \frac{1}{\sqrt{\pi}}\frac{(C_f)_{x=0}}{\lim_{x\to\infty}[\sqrt{x}\,C_f]} = \frac{1}{\sqrt{\pi}}\frac{\frac{1}{M_e}\sqrt{\frac{2}{\pi\gamma}}}{0.664\sqrt{\frac{\nu_e C}{u_e}}} \quad [5b]$$

Equation [5a] is the free molecule flow result for diffuse reflection (5). The denominator of Equation [5b] employs the Chapman-Rubesin (6) continuum result for wall shear. The latter assumes viscosity proportional to temperature according to the relation $(\mu/\mu_e) = C(T/T_e)$. Other choices for the free molecule and continuum shear could have been used. Substituting Equations [5] into Equation [3a] gives (for $\gamma = 1.4$ and $Re \equiv u_e x/\nu_e C$)

$$M_e C_f = 0.674\,\text{erfc}\left(0.573\frac{\sqrt{Re}}{M_e}\right)\exp\left(0.573\frac{\sqrt{Re}}{M_e}\right)^2 \quad [6]$$

Identifying $f(x)$ with q_w, as defined in Equation [2b], we get

$$a \equiv (q_w)_{x=0} = \frac{1}{M_e}\frac{\gamma-1}{\gamma}\sqrt{\frac{2}{\pi\gamma}}\left[1 + \frac{\gamma}{4}M_e^2 - \frac{T_w}{T_e}\right] \quad [7a]$$

$$b \equiv \frac{1}{\sqrt{\pi}}\frac{(q_w)_{x=0}}{\lim_{x\to\infty}[\sqrt{x}\,q_w]} = \frac{1}{\sqrt{\pi}}\frac{\frac{1}{M_e}\frac{\gamma-1}{\gamma}\sqrt{\frac{2}{\pi\gamma}}\left[1 + \frac{\gamma}{4}M_e^2 - \frac{T_w}{T_e}\right]}{0.410\left[1 + 0.424(\gamma-1)M_e^2 - \frac{T_w}{T_e}\right]\sqrt{\frac{\nu_e C}{u_e}}} \quad [7b]$$

Equation [7a] is the free molecule result for an accommodation coefficient of one (5) while the denominator of Equation [7b] employs the Chapman-Rubesin heat transfer result for $\sigma = 0.72$. If, in addition, $\gamma = 1.4$, the expression for local heat transfer becomes

$$\frac{M_e q_w}{1 + 0.350M_e^2 - \frac{T_w}{T_e}} = 0.193[\text{erfc}\,z]\exp(z^2) \quad [8]$$

where

$$z \equiv 0.265\frac{1 + 0.350M_e^2 - \frac{T_w}{T_e}}{1 + 0.170M_e^2 - \frac{T_w}{T_e}}\frac{\sqrt{Re}}{M_e}$$

In order for the expansions of Equations [3] to be valid, the coefficient of $\sqrt{Re}/M_e$, in Equation [8], must be positive. Thus it is required that

$$\frac{T_w}{T_e} > 1 + 0.350\,M_e^2 = F.\,M.\ \text{recovery temperature}$$

or

$$\frac{T_w}{T_e} < 1 + 0.170\,M_e^2 = \text{continuum recovery temperature}$$

For Equation [8] to give reasonable results, the ratio

$$[1 + 0.350\,M_e^2 - (T_w/T_e)]/[1 + 0.170\,M_e^2 - (T_w/T_e)]$$

should be of order one.

Equations [6 and 8], which are the main results of the present study, include effects of variable properties and dissipation since they were matched to continuum solutions which include these effects. The wall temperature is assumed constant in these equations. The overall friction and heat transfer coefficients for a plate of length L can be obtained by noting

The present approach results in relatively simple analytical expressions for wall shear and heat transfer on a semi-infinite flat plate. A major deficiency is that the low-speed Rayleigh problem model used here does not take into account boundary layer induced modifications of the free stream. For example, consider Equation [3c] to represent local shear in nearly continuum flow. Maslen (7) has shown that the first departure from flat plate continuum theory is associated with a self-induced pressure gradient which introduces a positive term of order $(1/b\sqrt{x})$ in the bracket of Equation [3c]. The present method therefore tends to underestimate the wall shear, at least in the border regime between continuum and slip flow. With regard to heat transfer, Maslen (8) has also shown that the departure from continuum first appears as a term of order $(1/b\sqrt{x})^2$ in Equation [3c] so that the present model has the correct order of magnitude with regard to the heat transfer perturbation in the border region between continuum and slip flow.

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Relaxation Lags in Gas Flow

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IT IS the purpose of this note to present a brief discussion of methods presently used to estimate lags in flow with

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consequent determination of the approach to equilibrium. One of the authors has recently developed a more precise treatment in which the detailed couplings of the flow equations are considered (1).[3] It is interesting to compare his results with those from more approximate considerations (2, 3).

In the case of vibrational temperature lag it is possible to separate a comparison factor J, which one may multiply with the more approximate result to achieve the more exact criterion. Following this procedure the vibrational lag in temperature for a diatomic molecule may be written as

$$\Delta T = (2/7)J\tau(DT/Dt) \quad [1]$$

where τ is the usual vibrational relaxation time, and DT/Dt represents the usual cooling rate. An outline of a derivation will be given later to show that

$$J = \frac{7(7 - 5K)}{2(K - 1)(7 + V^2/RT)} \quad [2]$$

Here, V is the flow velocity, R is the gas constant divided by the molecular weight and K is given as

$$K = \frac{7/2 + (\theta/T)^2/[\exp(\theta/T) - 1]^2}{5/2 + (\theta/T)^2/[\exp(\theta/T) - 1]^2} \quad [3]$$

where θ is the "vibrational temperature."

The factor J has been computed under various conditions for the molecule O_2, and the results are displayed in Fig. 1. From Equations [2 and 3] it may be observed that in the limit of infinite temperature J approaches 1. This yields the approximate formula[4]

$$\Delta T = (2/7)\tau(DT/Dt) \quad [4]$$

This formula when used at finite temperatures is a conservative estimate of the temperature lag; i.e., it gives a considerably larger value of the lag than when computations are made with the appropriate values of J for the temperature and velocity of interest. A temperature lag computed from [4] involves the following assumptions: (a) Complete vibrational partition (vibrational energy = RT) is attained, a condition only true at infinite temperature; and (b) there is a lag only in temperature, whereas actually there is a coupling in order to conserve energy such that there is also a lag in velocity. When the latter effect is considered, one obtains a velocity dependence as shown in Equation [2]. This dependence, which is small at the conditions for which the computations were made, vanishes at infinite temperature.

Equations [1 to 3] may be derived with the use of Equations [2C and 16] in (1), given below in modified form as Equations [5 to 7]

$$dT/T = E_1 dA/A \quad [5]$$

where

$$E_1 = (V^2/RT)(1 - K)/[(V^2/RT) - K] \quad [6]$$

and

$$(1/A)dA/dl = \Delta T/JTV\tau E_1 \quad [7]$$

where l is the length along a streamline. Equation [1] is seen to follow directly[5] from Equations [5 to 7] with the auxiliary use of the relationship

$$DT/Dt = VdT/dl \quad [8]$$

[3] Numbers in parentheses indicate References at end of paper.

[4] The formula used in (3), which neglected 2/7, results in an even more conservative estimate, as if all the energy change during flow came from the vibrational mode.

[5] The above derivation fails at the choking point because $dA = 0$; nevertheless, the same Equations [1 to 3] may be shown to follow from the equations valid at choking (1).

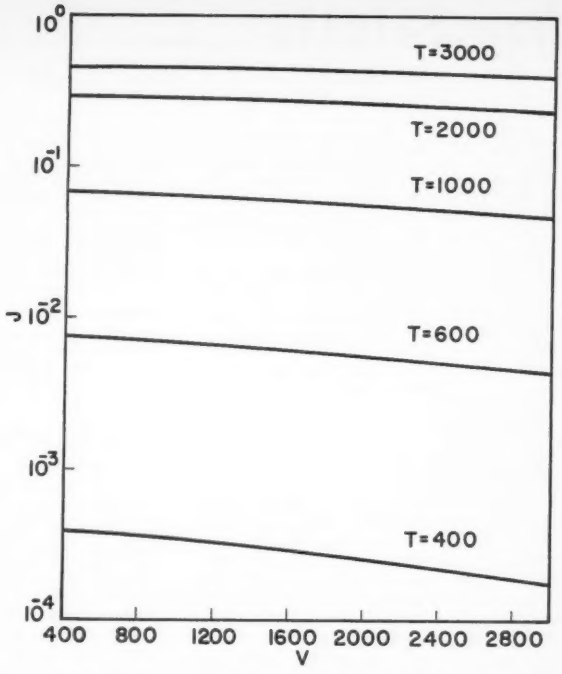

Fig. 1 J, the comparison factor, as a function of velocity in ft/sec for various absolute temperatures

which is true under steady state conditions.

It should be noted that according to (1) ΔT gives the value that the temperature lag approaches under the condition that $d(\Delta T/T)/dl \cong 0$. A better estimate for ΔT may be obtained from a determination of $d(\Delta T/T)/dl$ by the methods described in (1). It is therefore significant to state that if a *careful* measurement of rates, whether internal or chemical, is considered through the measurement of lags, one requires either a measurement which will yield the derivative along the streamline or measurements at two points so that the derivative may be computed.

In the case of chemical lag it has not been possible to separate a similar comparison factor, although it is possible to reduce the results in (1) to the form, $\Delta T = zDT/Dt$, which is the same form of the result in (2). Here z is the so-called reaction time. It would be likewise interesting to compare the results of calculations of z by the two different methods for a simple case as dissociation. In computing the reaction time by the method of (2) one neglects the following three effects, which are considered in (1). First, and probably most important, is that the temperature lag as defined in (2) would result if all the energy change during flow came from dissociation. This is analogous to the estimate for vibration given in (3).[6] Second, an analogous velocity effect would be presumed to be of the same order of magnitude as in the vibrational comparison given here. Third, there would be an added effect of density variation which does not appear in the case of vibration and is also probably small.

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[6] See note 4.

New Patents

George F. McLaughlin, *Contributor*

Optical tracking recording system (2,836,894). H. A. Wagner, Thousand Oaks, Calif., assignor to H. A. Wagner Co.

Tracking sighting means responsive to the movements of a control stick. Signals generated by the rate of motion of the control stick are recorded.

Fuel control for a gas turbine power plant (2,836,957). S. S. Fox, W. Hartford, Conn., assignor to United Aircraft Corp.

Metering valve in the conduit upstream of a chamber through which the fuel flows, feeds maximum fuel into the chamber as a function of turbine speed, compressor air inlet temperature and compressor discharge pressure.

Jet power plant with unobstructed rotating combustion chamber (2,836,958). J. A. Ward III, Duluth, Minn.

Rotary jet engine with front and rear supports having axially aligned bearings in which hubs rotate. Gases flow through the engine in a straight path, and a stationary fuel supply means extends into the combustion chamber through one of the hubs.

Gas turbine engine supporting frame (2,836,959). C. J. McDowell and O. V. Montieth, Indianapolis, Ind., assignors to General Motors Corp.

Rotary engine with a cylindrical support secured to a torque ring, and extending rearward. A central shaft has a rear bearing connected by rear struts to the cylindrical support.

Wing mounted jet propulsion system (2,838,256). I. M. Davidson and N. A. Dimmock, Woking, England, assignors to Power Jets (Research and Development), Ltd.

Hollow wing with an air inlet extending along the leading edge. A hollow main spar is formed with inlet and outlet apertures. Jet engines are mounted within the wing supplied by air from air supply passages and discharging in narrow exhaust slots along the wing trailing edge.

Jet sustained aircraft with enclosed compressor rotor (2,838,257). M. Wibault, New York, N. Y., assignor to Vibrane Corp.

Centrifugal blower in the body of an aircraft. The downstream end of the outlet passage through which air is discharged from the blower discharges into the ambient atmosphere in which the aircraft flies, and different parts of the downstream end are at different locations with respect to the c.g. A control varies the amount of air discharged at the different locations.

Device for demonstrating mathematical probability (2,838,851). R. Lusser, S. Pasadena, Calif.

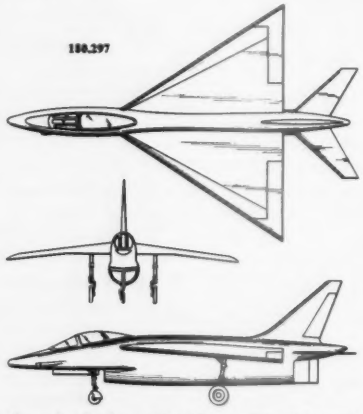
Group of balls each movable with respect to all remaining balls, and composed of electrically conductive material, except that a predetermined area (constituting a minor fraction of the surface of some of the balls) is nonconducting. An electrical circuit in each ball is completed by contact with the conducting surface of each ball. Movement results in a random rearrangement of the balls, breaking the circuit when contact with any one or more balls lies within a nonconducting surface area.

Means for the deflection of fluid jets (2,838,909). H. L. P. Meulien, Courbevoie, France, assignor to SNECMA.

Pair of movable members mounted for angular displacement in the exhaust nozzle. The members are streamlined when closed, allowing undisturbed flow of the jet discharge. When opened, they expose two intercepting surfaces to deflect the flow.

Sun follower (2,839,689). G. M. Trinite Jr., Baltimore, Md., assignor to Aircraft Armaments, Inc.

Body containing coarse-eye photoelectric cells responsive to light rays over a wide sector or space to produce an output voltage. A plurality of fine-eye cells responsive to rays within a relatively narrow sector of space within the wide sector, to produce an output voltage. The fine-eye cells alone provide output while the sun follower is closely tracking the sun.



Aircraft design (180,297). E. Schmued and W. E. Gasich, Los Angeles, Calif. assignors to Northrup Aircraft, Inc.

Delta wing jet airplane with air intake scoop below fuselage, aft of cockpit. The tail is at the same level as the wing.

Variable area jet nozzles for gas turbine units (2,839,890). F. W. W. Morley, Castle Conington, England, assignor to Rolls-Royce, Ltd.

Jet pipe with an air inlet leading to the space between inner and outer skin members. An outlet in the nozzle induces the flow of cooling air through the air passage. Nozzles may be adjusted to vary the orifice area.

Thrust reversers for jet engines (2,839,891). G. T. Drakeley, Seattle, Wash., assignor to Boeing Airplane Co.

Deflector flaps mounted for movement between an inoperative position along the tailpipe and an operative position to the rear of (and directed outwardly from) the tailpipe axis. In operative position, gases are directed forward as they follow the interior surface of the flaps.

Gas Turbine cycle (2,839,892). H. Rosendahl, Yonkers, N. Y.

Process of making valuable products of incomplete oxidation of a fluid-fuel by employing secondary fuel as a coolant and utilizing the turbine exhaust gases in chemical reactions. Turbine gases are passed into a chamber in the presence of a catalyst, using the sensible heat of the turbine exhaust for initiating catalytic reaction.

Multiple vortex tube generator cooling unit (2,839,898). F. H. Green, Los Angeles, Calif., assignor to The Garrett Corp.

System for cooling and pressurizing an instrumentality in an aircraft having a jet engine compressor and a ram air duct. Flow of high pressure air controlled through vortex tubes is limited when sonic velocity of air is reached.

Regenerative vortex cooling systems (2,839,900). F. H. Green, Palos Verdes Estates, Calif., assignor to The Garrett Corp.

System for supplying cooling air to an aircraft. Means for passing ram air in heat exchange relation to first and second vortex tubes.

Position indicator (2,839,921). A. Wood, Albuquerque, N. M. assignor to Engineering Development Co., Inc.

Rack representing the equator, in engagement with arms on a pivot representing the elevated pole. A track guide represents departure and destination positions, and a movable guide corresponds to estimated ground speed. A gyro compass having latitude correction is connected to a latitude gear engaged to the rack and arms.

Shaped charges (2,839,997). J. H. Church and G. J. Kessenich, Madison, Wisc., assignors to the U. S. Army.

Pair of identical six-pointed star-shaped sheets of material having uniformly spaced arms. Sheets are connected parallel axially, and a strip of material extends along the edges of the sheets to form a star-shaped container which is filled with an explosive charge.

Air intake for aircraft power plants (2,840,322). A. A. Griffith, Derby, England, assignor to Rolls-Royce, Ltd.

Center body dividing an air intake into two passages. The body has a wedge shaped so that at the design Mach number the shock wave formations extend outside or onto the leading edge lips. Intake passages may be varied in effective area.

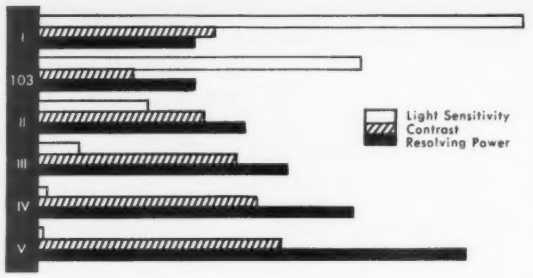
Hinged airfoil with jet nozzle for aircraft flight control (2,840,323). G. K. Hood Jr. and J. A. O'Malley Jr., N. Tonawanda N. Y., assignors to Bell Aircraft Corp.

Attitude control consisting of a pair of telescoping cylindrical members rotatable about the longitudinal axis. One member is mounted on a fixed airfoil and the other on a movable airfoil. A duct conveys pressurized gas into the inner member so that upon swinging the movable airfoil away from its neutral position, parts align to release the gas into the atmosphere, rotating the airplane about a flight control axis.

Aircraft structure (2,840,325). A. A. Griffith, Derby, England, assignor to Rolls-Royce, Ltd.

Pilot's windshield with an external surface lying in the surface of a cone intersecting the external skin, and disposed with its apex forward. At supersonic speed a shock wave formation is generated which is of conical surface form with its apex coincident with the apex of the windshield.

EDITOR'S NOTE: Patents listed above were selected from the Official Gazette of the U.S. Patent Office. Printed copies of patents may be obtained from the Commissioner of Patents, Washington 25, D. C., at a cost of 25 cents each; design patents, 10 cents.



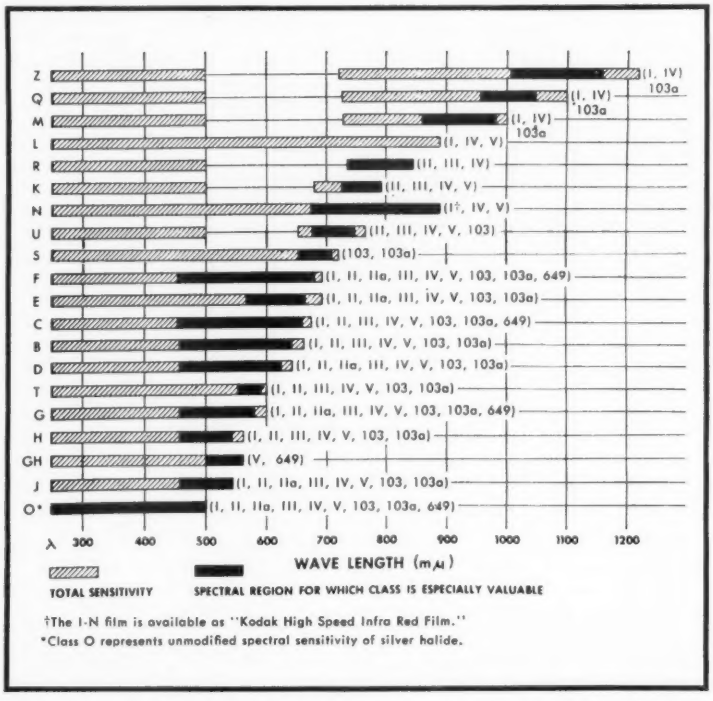
Emulsion Types: *Type I* has the highest order of light sensitivity provided by modern emulsion-making methods, medium contrast (gamma somewhat greater than 1), and resolving power of about 60 lines per millimeter for a test pattern of 20:1 contrast. With this as a basis, the chart shows how *Types I, 103, II, III, IV,* and *V* differ. In addition there are:

Type 649—Extreme resolving power.

Types IIa and 103a—Similar to II and 103 respectively, but much higher in sensitivity for low-intensity, long-time exposure.

This chart shows the approximate relationship between the principal emulsion types with respect to light sensitivity, inherent contrast, and resolving power.

We hope these charts impress you



This chart shows the spectral sensitivity regions of the various classes of sensitizing and the emulsion types in which they can be furnished.

In the more advanced aspects of photography, more ideas are floating around today than can be prudently enumerated in a freely circulating journal. Suddenly the potentialities of the photographic emulsion have become highly exciting. Suddenly many people whose interests hitherto have been concentrated on other fields have to learn the most they can about photography in the least time.

First off, they should familiarize themselves with the two charts above. The charts illustrate how specifically we can "tune" a photographic emulsion to its purpose. Many years ago, the astronomers (and others who have to define photographic characteristics with high precision) came and told us their needs. We responded with a large family of emulsions that has expanded and greatly improved over the years. Today there are 111 of these "Kodak Spectroscopic" emulsions, and practically every astronomer knows to order by a combination of a numeral and a letter, as explained in the above charts, e.g. I-C, IV-L, 103a-Z.

Maybe one of these combinations just neatly fits one of the brave new ideas now being born. Maybe something is required that isn't in the charts yet. Maybe it can be made, and maybe it can't. What needs to be impressed on the authors of the ideas and their principal helpers is that very early in their photographic thinking they ought to address a few questions to:

Special Sensitized Products Sales

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Book Reviews

Ali Bulent Cambel, Northwestern University, *Associate Editor*

Elements of Pure and Applied Mathematics, by Harry Lass, McGraw-Hill, New York, 1957, 591 + ix pp. $7.50.

Reviewed by PATRICK BENSON
TEMPO, General Electric Company

This book is a mathematical compendium comparable to those of Reddick and Miller, Margenau and Murphy, Pipes, Wilie, etc.; and, like those, it has certain merits and defects. In this case, however, it appears that the former far outweigh the latter. The book is well named. Dr. Lass sticks to the elements of mathematics; at the same time, however, he provides a number of practical examples which will keep engineers and physicists interested in the material. Furthermore, his choice of examples is timely. For example, he includes practical discussions of pursuit problems; topics from game theory, group theory; applications of the Laplace transform, nonlinear mechanics, and so on, all of which are becoming more and more important in today's technology.

An advantage of the book is the number of examples worked out, all of which are wisely chosen. While, additionally, a set of problems is given at the end of each chapter, answers to these sets are not provided. The present reviewer feels that one of the more important functions of a good compendium of this type is its use as a reference text or a self-study text. In either case a list of answers is an important asset.

To take the work in chapter order, the first deals with determinants, linear equations and matrices. In the second chapter, the essentials of vector analysis are covered. Here Dr. Lass includes the discussion of the kinematics of a rigid body with one fixed point, and the discussion of pursuit problems. Chapter 3, Tensor Analysis, starts with the concept of generalizing the notion of a vector and leads into a standard discussion of tensors. Some of the applications include Riemannian geometry, Hamilton's equations of motion and the Navier-Stokes equation of hydrodynamics. The second and third chapters are unusually well presented.

In Chapter 4, those elements of complex variable theory are included which would normally be encountered in an elementary senior or first year graduate course, but there is greater emphasis on rigorous proof than is usually found in textbooks of this type. Many of the problems are important in engineering. In particular the application of the Schwartz-Christoffel transformation to electrostatic problems is included.

In Chapter 5 topics of ordinary and partial differential equations are treated and some of the typical solution functions (e.g., hypergeometric, Laguerre and Hermite polynomials) are introduced. Unfortunately, a more complete discussion of Eigenvalues and Eigenfunctions, which could have been related to the solutions of boundary value problems, is omitted. They are mentioned only briefly at the beginning of the book. Chapter 6 includes the elements of orthogonal polynomials, Fourier series and Fourier Integrals. The author points out that these, combined with the discussions under complex variables and the calculus of variations, usually provide sufficient background for the student to understand Wiener's results concerning linear predictors and filters. Some discussion of nonlinear differential equations involving use of Fourier series is also included.

In Chapter 7 he deals with the Stieltjes Integral, the Laplace transform and the calculus of variations. The treatment is brief but there seems to be enough material to enable the student to understand applications to the problems of physics and engineering.

In Chapter 8 two of the more fundamental, and consequently much more difficult, topics of mathematics are included, namely group theory and algebraic equations.

Chapter 9 contains discussions of the elementary concepts of probability theory, the chi-squared distributions of statistics, and a brief discussion of game theory and the Monte Carlo method. Since it is becoming more apparent than ever that the engineer must have a thorough knowledge of probability theory if he is to keep abreast of such fields as information theory and filter theory, this is an important contribution in a textbook.

The final chapter of the text deals with many basic aspects of real variable theory. It is over 100 pages long and, taken by itself, could comprise a text on real variable theory.

In general the text is written in such a way that each chapter is independent of the others, although there are several cross references. The reader who is interested chiefly in obtaining a cursory knowledge of the material will probably find that the text contains most of the elementary topics found in existing applied mathematics texts. On the other hand a sufficient degree of rigor is introduced to make the treatment more palatable to the pure mathematician.

Chemistry Problems in Jet Propulsion, by S. S. Penner, Pergamon Press, New York, 1957, 394 + xiv pp. $12.50.

Reviewed by JOHN B. FENN
Project SQUID, Princeton University

It is a long way from the principles which put an electron into an orbit of the Bohr atom to the practice which pushed a satellite into an orbit of Earth. Professor Penner has written an excellent guidebook for this ambitious journey in chemical science and technology. The trip has been broken into three stages. The first is brisk passage through the development of modern structural chemistry. It begins with spectroscopic observations. By way of the elements of quantum theory it arrives at a description of chemical bonds and their implications as energy sources and sinks. This is home territory for the author. His familiarity with the terrain is apparent. The traveler's road is smooth and clearly marked except for the last few steps. Compared to the highway which precedes it, the sketchy account in Chapter VII of chemical compounds as propellants is rather like a detour.

The second stage of the journey comprises a tour through classical thermodynamics and thermochemistry. It begins with a briefing on the mathematics of partial differentiation and the properties of differentials. It includes instructions for computing equilibrium temperatures and compositions under both static and flow conditions. It concludes with a treatment of the calculation of thermodynamic properties of gases from structural and spectroscopic data, where again, with evident relish, the author is in his element.

The last lap of the journey leads through the relatively unknown territory of rate processes. After an introduction to classical chemical reaction kinetics, conservation principles in reacting flow systems and transport processes in gases, a number of problems are treated in detail. These include diffusion flames, premixed laminar flames and reactions during expanding flow. Following a discussion of similarity and dimensional analysis, the author finally considers some of the real and complicated technical problems in liquid propellant rockets which have not succumbed to exact analysis. There are short stopovers at combustion instability, scaling, transient phenomena, carbon formation, combustion of sprays, liquid monopropellant burning and combustion volume requirements.

Thanks to his terse and lucid style, Professor Penner has been able to cram a surprising amount of information into this relatively small guidebook. The numerous and extensive tabulations of useful data are extremely welcome. The cited references are adequate charts for excursions from the main route for more detailed views of the scenery along the way. Be he an eager student or jaded expert, the traveler cannot fail to be stimulated into the realization that there is still much country to be settled and many new trails to blaze.

On the other hand, the chosen route bypassed many attractions. This tourist was disappointed at the absence of an account of many of the interesting problems relating to solid propellants. He regrets the omission of any treatment of premixed turbulent flames or homogeneous combustion as in the stirred reactor concept. The discussions of detonation and heterogeneous kinetics were tantalizingly brief. The philosophy which justified rather extensive review of quantum theory could surely have allowed a section on the elements of the kinetic theory of gases. To keep a book within reasonable size or a course within reasonable scope, an author and teacher is forced to be selective. His choice of topics is a matter of taste to which there can be little objection. However, the text contains so much which is so well done that one cannot help but

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BMEWS—Ballistic Missile Early Warning System—is under development to provide a scientific answer to the detection of intercontinental ballistic missiles. In its various functions, it will be one of the electronic wonders of the age. The unblinking eyes of its strategically located radars are being developed to detect an onrushing missile thousands of miles away. Almost at once electronic computers will determine altitude, course and speed, and set in motion the necessary defense apparatus. RCA acknowledges its tremendous responsibility as prime contractor for the design and construction of BMEWS—so vital to our country's defense and so effective as an instrument for peace.

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wish there had been more.

Of more concern than omissions are the implications of this book in terms of a growing problem in engineering education. It is food for thought that incumbent upon a graduate course in aeronautical engineering is the need to cover elementary quantum mechanics, thermodynamics and even some basic mathematics. At least half of the book, were it separated from the title page and preface, could never be identified by a casual observer as jet propulsion course material. Familiarity with quantum theory and thermodynamics is fully as important to other kinds of engineers and scientists. One can reasonably ask whether a student who does not have a working acquaintance with these basic disciplines is in any position to have already decided to concentrate on jet propulsion or any other specialized technology. By the same token it would seem a pity that Professor Penner's admirable presentation of this fundamental material should be available only to engineering students majoring in jet propulsion.

Vibration and Impact, by Ralph Burton, Addison-Wesley Publishing Co., Inc., Mass., 1958, 310 + x pp. $8.50.

Reviewed by D. E. HUDSON
California Institute of Technology

The two classic engineering vibration books of Timoshenko (1928) and Den Hartog (1934) established a pattern which has well served the student and the teacher. Through successive revisions these books have increased their usefulness as standard texts and references in the field. Recent developments, however, in both the subject matter itself and in its place in engineering education have provided incentives for a number of new books with significantly different approaches. The present book is an example of one type of response to these new requirements.

The major principles behind this book appear to be: (1) To provide an elementary treatment of the subject suitable for undergraduate students with a minimum background of dynamics and mathematics; (2) to emphasize the physical aspects rather than the mathematical, and the applications rather than the basic theory; (3) to include brief introductory treatments of a number of topics of current interest, such as shock problems, random loads, control systems, instrumentation and acoustics.

The book is designed as a textbook; many of the explanations would require a considerable expansion by the teacher, and a number of the problems would need much class discussion. These are of course advantageous features from the teaching standpoint, provided that close class supervision is available.

The laudable desire to introduce many topics of current interest sometimes leads the author to oversimplifications. The treatment of random loads, for example, seems inadequate even as an introduction, and the amount of attention given to the impact problem hardly justifies the prominence of the word in the title. In some cases, as in the group of formulas for drag forces, the restrictions on the use of the information have not been fully stated, and some of the examples of damping calculations seem to imply an ease and exactness of calculation that might not be realized in practice. The definition of equivalent viscous friction used by the author at one point seems to be a different one than customarily used, and for hysteresis damping it is not clear that the results obtained by the two approaches are similar.

A commendable feature of the book is the introduction and frequent use of the concept of dynamic stability, and this enables the author to show the unity between vibration problems and control system problems. Other favorable features include the interesting brief historical introduction and the annotated reference lists.

Mechanical Vibration courses are now quite common in our undergraduate engineering curricula, and this book may find a useful place there for the brief introductory course. The student who wishes ultimately to fully develop his knowledge of the field, however, would probably be better advised in his undergraduate program to concentrate on basic dynamics and mathematics, which he could then use as a foundation for advanced studies.

Book Notices

Nuclear Engineering, by C. F. Bonilla, McGraw-Hill, New York, 1957, 850 pp. $12.50. This excellent text was prepared to give the engineer the details of nuclear powerplant design. Because of the vastness of the subject matter, different outstanding specialists wrote about their respective fields. The authors and their subjects are:

Theodore Baumeister—Power Generation
Charles F. Bonilla—The Flow of Fluids; Heat Removal
John R. Dunning—Introduction
Gioacchino Failla—Basic Concepts of Radiation Protection
Alfred M. Freudenthal—Thermal-Stress Analysis and Mechanical Design
William W. Havens Jr.—Nuclear Particles; Nuclear Physics
John W. Hoopes Jr.—Instrumentation and Control
R. Wayne Houston—Introduction; Elementary Reactor Physics
George L. Kehl—Metallurgy of Uranium and Uranium Alloys
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David C. Peaslee—Shielding of Power Reactors
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The First One Hundred and Fifty Years, by The House of Wiley, John Wiley & Sons, 1957, 242 pp. $7.50. An interesting book relating the history of the John Wiley & Sons, Inc., publishing house between the years 1807–1957 and written by many leaders from industrial firms and educational institutions.

The Cosmic Radiation, by J. E. Hooper and M. Scharff, Wiley, New York, 1958, 172 pp. This little volume should be useful to those who wish to get a bird's eye view of the cosmic radiation techniques for measurements and physical effects. It is not a popular book, but requires a good background in modern physics including some nuclear physics.

Technical Literature Digest

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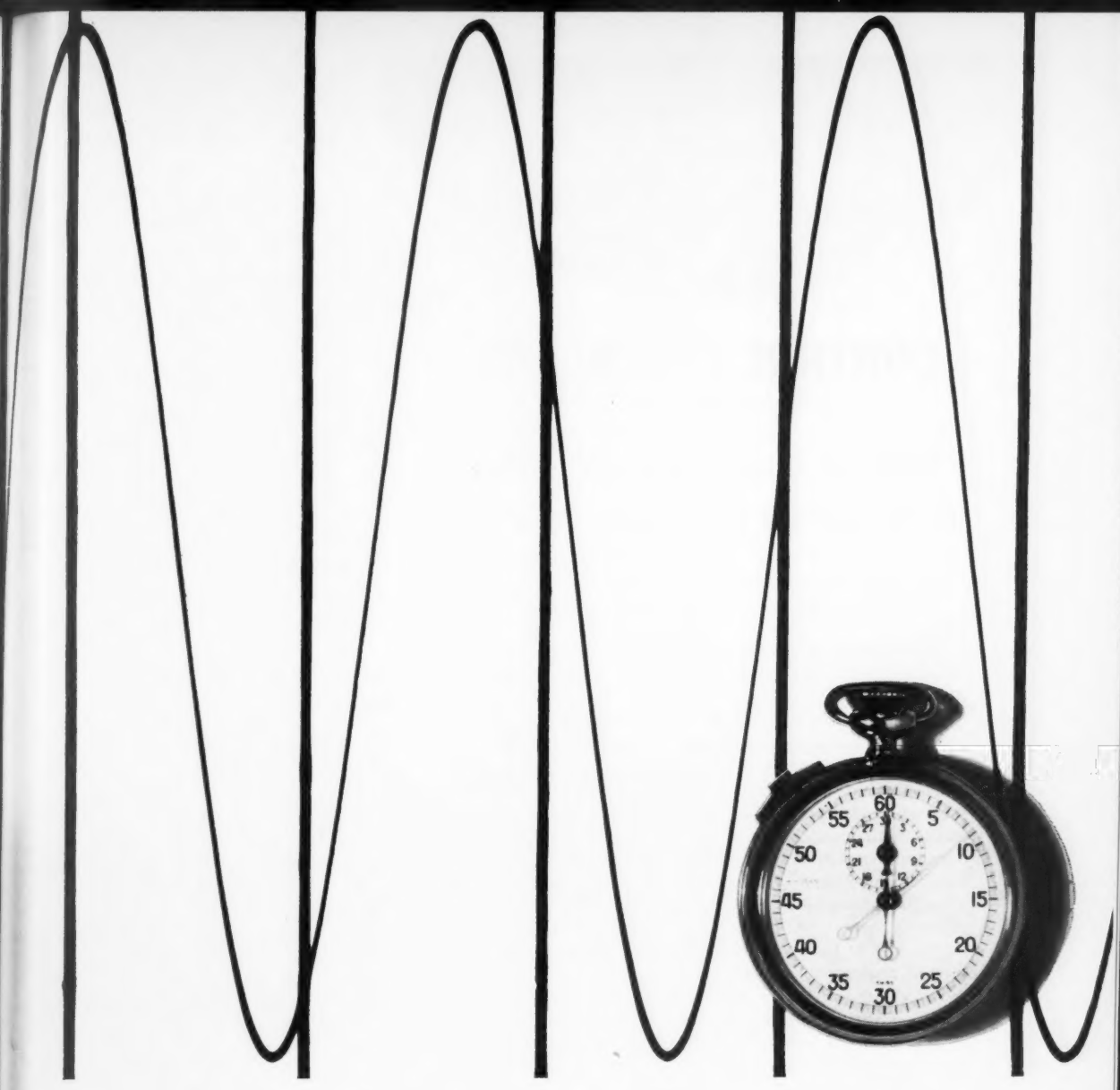
EDITOR'S NOTE: Contributions from E. R. G. Eckert, J. P. Hartnett, T. F. Irvine Jr. and P. J. Schneider of the Heat Transfer Laboratory, University of Minnesota, are gratefully acknowledged.



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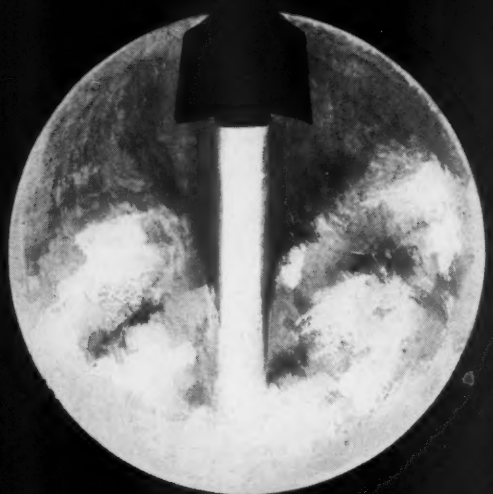

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